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CONNECTION OF CLOSE QUARTERS TO GENERALIZED TURNING POINTS.(U)
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ABSTRACT

The solutions of Schroedinger equations

$$\varepsilon^2 d^2w/dz^2 + q^2w = 0, \quad q = q(z), \ \varepsilon \to 0 \tag{1.1}$$

have well-known WKB-approximations, but the coefficients in these differ on the two sides of a turning point. A new method for connecting them across such points is developed to extend present theory to a more general class of turning points, which includes logarithmic branch points of $q(z)$, among many others. To this end, a delicate contraction for an integral equation differing from those of Langer and Olver is used to show that Bessel functions can still approximate the solutions at a certain, small distance from the irregular point of (1.1), even though not uniformly near it. A novel feature of the analysis is that the extreme variation of the exponential kernel is here controlled even on non-progressive paths. Connection is completed radially by means of the same integral equation.

AMS (MOS) Subject Classification: 34E20, 41A60, 30E15
Key Words: Schrodinger equation, oscillator modulation, WKB-connection, turning point
Work Unit Number 1 (Applied Analysis)

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SIGNIFICANCE AND EXPLANATION

This work concerns the modulation of waves or oscillating systems, which pervade all the science and engineering disciplines. Modulation occurs when waves travel through an inhomogeneous material in which the local propagation velocity differs from place to place, but the differences are small over a distance of only a wavelength -- a very common case in the sciences and engineering. The resulting change to the waves is mostly gradual, but occasionally drastic, as at a shadow-boundary, where oscillation turns into decay and quiescence over just a few wavelengths. When this phenomenon can be analyzed via an ordinary differential equation, such a boundary is called a turning point.

At first, only the simplest turning points representing the most typical shadow boundaries were studied. But then some phenomena, such as wave reflection and scattering cross-sections, came to be traced to hidden turning points that become visible only when real distance (or time) is embedded in its complex plane. When the material properties vary in a general manner, (which can often be observed only incompletely) the hidden turning points can have arbitrarily complex structure. The following work extends the basic mathematical formula for connecting waves with shadow across a turning-point boundary to a much larger class of variations in the material properties than had been accessible up to now.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC and not with the authors of this report.

CONNECTION OF CLOSE QUARTERS TO GENERALIZED TURNING POINTS

J. F. Painter[†] and R. E. Meyer

1. Introduction.

The best-known and most important approximate solution to the equation

$$(1.1) \qquad \varepsilon^2 d^2 w/dz^2 + q^2(z)w(z) = 0 , \qquad \varepsilon \to 0 .$$

is the WKBJ solution, sometimes also called the Liouville-Green solution,

$$(1.2) \qquad w(z) \sim A q^{-1/2}(z) \exp[i\varepsilon^{-1} \int^z q(z')\,dz'] + B q^{-1/2}(z) \exp[-i\varepsilon^{-1} \int^z q(z')\,dz'] .$$

This solution has only limited value around a singularity of q'/q. It is valid in a closed region R which has no singularities or zeroes of q and which satisfies a certain convexity property. Because of this convexity property, such a region R can contain only one "side" of a singularity or zero of q. In practice, one needs to know $w(z)$ on both sides at once, and that requires a much more careful study of $w(z)$ than needed to obtain (1.2).

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This convexity property is defined in terms of the natural independent variable ξ, which appears in (1.2):

$$\xi = \int_0^z q(z')dz' \quad (1.3)$$

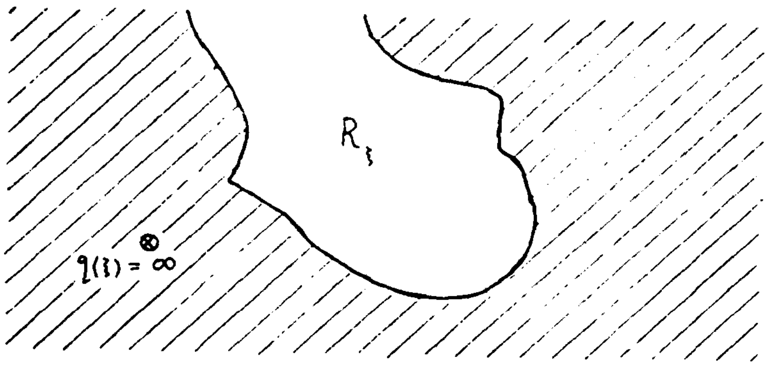
Consider $R = R_\xi$, the region of validity of (1.2), as a set in the ξ-plane. If two points of R_ξ have the same imaginary part, then the horizontal line joining them must itself lie in R_ξ. As R_ξ does not contain any singularity or zero of q, this means that it cannot have any ξ both to the right and to the left of a singularity or zero of q (Figure 1.1). The need for this convexity property arises in the proof of (1.2) from a need to have the exponentials monotone on curves connecting points of R_ξ, so that they can be bounded in terms of their magnitude at endpoints.

The limitation on where (1.2) can be applied leads to the connection problem: given the values of the coefficients A,B for which (1.2) applies on the left side of $z = 0$, a singularity or zero of q, find their values on the right side.

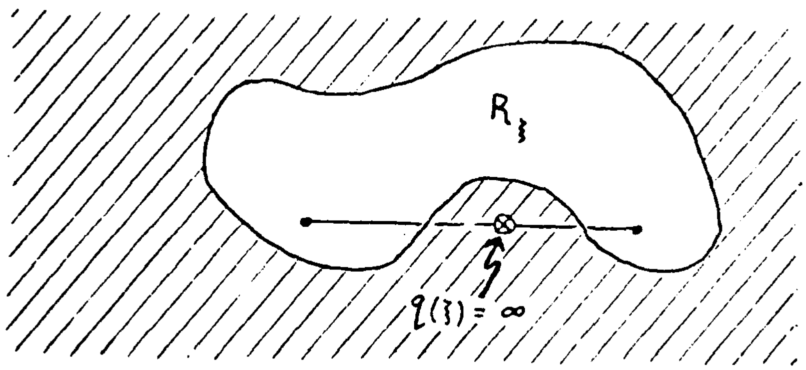
The most satisfactory way to solve this problem would be to get a closer approximation to $w(z)$ than (1.2) - one that is good at both sides of $z = 0$ simultaneously. Langer (1931, 1932, 1935), Riekstins (1958), and Olver (1977) have done exactly that for the simplest case, where $q(z)$ is approximately a real power of z near $z = 0$. Then the approximation (1.2) by exponentials may be replaced by an approximation by Hankel functions. The exponential approximation

FIGURE 1.1

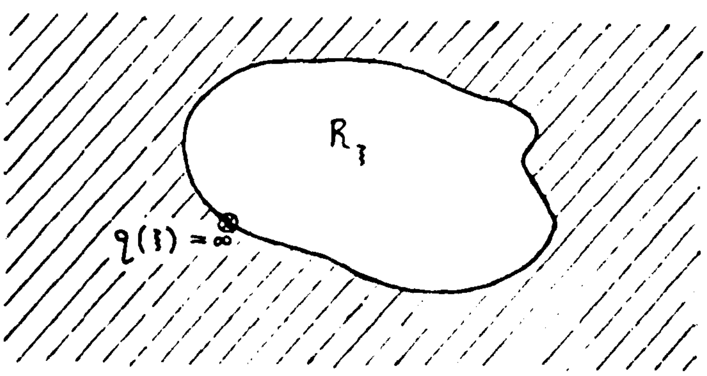
(a) A possible region R_ξ where (1.2) can apply



(b) A region in which (1.2) is not applied



(c) A region in which (1.2) is not applied



(1.2) holds in a set bounded away from $z = 0$, while the Hankel function approximation holds arbitrarily near $z = 0$ as well.

When $q(z)$ is not a power of z, the connection problem grows more complicated. Painter (1979) has shown for a logarithmic $q(z)$ and imaginary ε that the Hankel functions will not serve any better than the exponentials of (1.2). The central connection methods which work when $q(z)$ is a power of z are not adequate in greater generality because the local approximands are much less tractable.

The problem is that the singularity of $q(z)$ at $z = 0$ may, in the general case, be vicious enough to upset computations made near $z = 0$. The most natural way to deal with this problem is to run away from it, to solve the connection problem as a "lateral" connection problem, solving the equation at $|z| = \infty$.

This sort of method, however, is doomed from the start. Distance from the singularity makes it hard to use information about it, but it turns out that the answer to the connection problem is strongly dependent upon the nature of the singularity at $z = 0$.

The route to the solution of this dilemma lies in the middle of the road, between the central and lateral approaches. Equation (1.1) will be solved, approximately, at a carefully controlled distance from $z = 0$. Some of the advantages of both the central and the lateral connection approaches will show up well enough to give an answer to the connection problem.

The method used will be to derive, from the differential equation (1.1), an integral equation for the coefficients $A(\xi)$, $B(\xi)$ for which the approximate solution (1.2) is exact at z. That is,

$$(1.3) \qquad w(z) = A(\xi)q^{-1/2}(z)\exp(i\xi/\varepsilon) + B(\xi)q^{-1/2}(z)\exp(-i\xi/\varepsilon) .$$

The integral equation will then be written into forms for which it can be partially solved by the contraction mapping theorem, i.e., by iterations. Some of these versions of the integral equation give $A(\xi)$ or $B(\xi)$ in various regions; others show that $A(\xi) - A^{\gamma}(\xi)$ is small in certain areas, where A^{γ} is the result of substituting z^{ν} for $q(z)$ and can be expressed in terms of Hankel functions. Proving that the iterations which solve the equations work, involves many estimates, which depend especially upon careful choices of the curves along which various integrals are computed.

We will be studying the connection problem for a class of singularity at $z = 0$ defined in terms of a modulation function $\varphi(\xi) = \frac{1}{2}q'(z)/q^2(z)$ by $\varphi(\xi) \sim \gamma/\xi$ near zero, where γ is any real constant $< \frac{1}{2}$. In terms of q, this includes the case which Langer and others have studied, where $q(z) \sim z^{\nu}$ as $z \to 0$. More important, it generalizes that case to include logarithmic "turning points" such as $q(z) \sim z^{\nu}(\log z)^{\mu}(\nu > -1)$ and many others. It will be proven that the same connection formula which holds for $q(z) \sim z^{\nu}$ also holds for this general case; that is, when the region of interest connects regions to left and right above the turning point (Fig. 1.1b), the coefficients A_r, B_r in (1.2) to the right of zero depend upon those on the left as follows:

$$A_r = A_\ell - 2B_\ell i \sin(\gamma\pi) + o(1) \tag{1.4}$$
$$B_r = B_\ell \qquad\qquad + o(1) .$$

A dual formula, in which A and B exchange roles, applies when the connection is made below the turning point (Olver 1974). Many other formulations of the WKB connection problem arise in practice, but all are reducible straightforwardly to (1.4) or its dual.

Section 2 is devoted to the formulation of the integral equation which was found capable of solving the connection problem with more generality than before. In some ways, it is intermediate between the simple WKB integral equation with exponential kernel (Olver 1974) and Langer's, with Bessel kernel. The properties of the new kernel, which is related to the incomplete Gamma function, are discussed in Section 4. The main Section 3 outlines the method of connection along a semi-circle of radius $\delta(\varepsilon)$ tending to zero almost as fast as ε; Section 5 supplies details of the proofs. In practice, connection between points left and right, but so close to the turning point, is inadequate and Section 6 therefore extends the results to arbitrarily large distances.

While our proof of (1.4) is valid also for $\gamma = 0$, the formula is not very informative in that case. The reason is that no information has then been specified about $\varphi(\xi)$. Modulation functions behaving like $(\xi \log \xi)^{-1}$ near zero have been treated by Painter (1979) along lines paralleling those here reported, but involving much added labor because the kernel is more complicated. More definite results than (1.4) were deduced (Painter 1979) for that class, which includes, e.g., $q(z)$ which are, or approximate, a power of log z near $z = 0$.

2. Integral Equations for Connection.

The main integral equations needed to solve the connection problem come from the original differential equation, (1.1), which we take to be valid for z in an open set R_z of complex numbers. We assume $q(z)$ to be holomorphic and nonzero in R_z; if q have a singularity or zero, it will lie on the boundary of R_z. We take ε to be positive.

The WKB solution suggests that we look at (1.1) in a different form. This approximate solution is

$$(2.1) \qquad w(z) \sim Aq^{-1/2}(z)\exp[i\xi/\varepsilon] + Bq^{-1/2}(z)\exp[-i\xi/\varepsilon]$$

$$(2.2) \qquad w'(z) \sim (iA/\varepsilon)q^{1/2}\exp[i\xi/\varepsilon] - (iB/\varepsilon)q^{1/2}\exp[-i\xi/\varepsilon]$$

where A and B are constants and

$$(2.3) \qquad \xi = \xi(z) = \int_{z_0}^{z} q(z')dz' .$$

With no loss of generality, we may take the fixed point $z_0 = 0$.

The approximation (2.1), (2.2) is valid in closed bounded subsets S_z of R_z with sufficiently smooth boundaries, which satisfy a "horizontal convexity" property. If we set $S_\xi = \xi(S_z)$ then S_ξ is horizontally convex iff for any two points of S_ξ with equal imaginary parts, the line joining them also lies in S_ξ. Because the variable ξ shows up in a more fundamental way than does z here, ξ is a more natural independent variable.

This horizontal convexity requirement, moreover, is the source of the connection problem. Suppose that $q(z)$ has a possible singularity or zero at one point of the boundary of R_z, say at $z = 0$. (Here and henceforth we assume that $q(z)$ is holomorphic and nonzero at the other boundary points of R_z). We must assume $q(z)$ to be integrable at $z = 0$ so ξ can be defined by (2.3). We may use (2.1) to compute w in a large neighborhood of a point to the left of zero, $\xi_\ell < 0$, and in a neighborhood of a point to the right of zero, $\xi_r > 0$; nevertheless if we were to know the coefficients A_ℓ, B_ℓ for which (2.1) holds near ξ_ℓ, equation (2.1) would not help us find the correspondin- coefficients A_r, B_r for ξ_r. This is because (2.1) cannot be applied in a set which contains both ξ_ℓ and ξ_r at once.

For the connection problem we seek to know A_r and B_r in terms of A_ℓ and B_ℓ. We will find A_r and B_r by considering (2.1), (2.2) as equations exactly true throughout R_z, that is,

$$q^{1/2}w(z) = A(\xi)\exp(i\xi/\varepsilon) + B(\xi)\exp(-i\xi/\varepsilon)$$

$$\text{(2.4)} \qquad \varepsilon w'(z)/(iq^{1/2}) = A(\xi)\exp(i\xi/\varepsilon) - B(\xi)\exp(-i\xi/\varepsilon)$$

$$\text{for} \quad \xi \in R_\xi = \xi(R_z) \ .$$

Thus, $A_\ell = A(\xi_\ell)$, $B_\ell = B(\xi_\ell)$, $A_r = A(\xi_r)$, and $B_r = B(\xi_r)$.

Substitution of (2.4) into (1.1) yields equations for the modulation coefficients $A(\xi)$ and $B(\xi)$,

$$dA/d\xi = \varphi(\xi)B(\xi)e^{-\rho\xi} \tag{2.5}$$

$$dB/d\xi = \varphi(\xi)A(\xi)e^{\rho\xi} \tag{2.6}$$

where φ is the "modulation function"

$$\varphi = \frac{1}{2}q^{-2}\,dq/dz = \frac{1}{2}\,q^{-1}\,dq/d\xi \tag{2.7}$$

and $\rho = 2i/\varepsilon$.

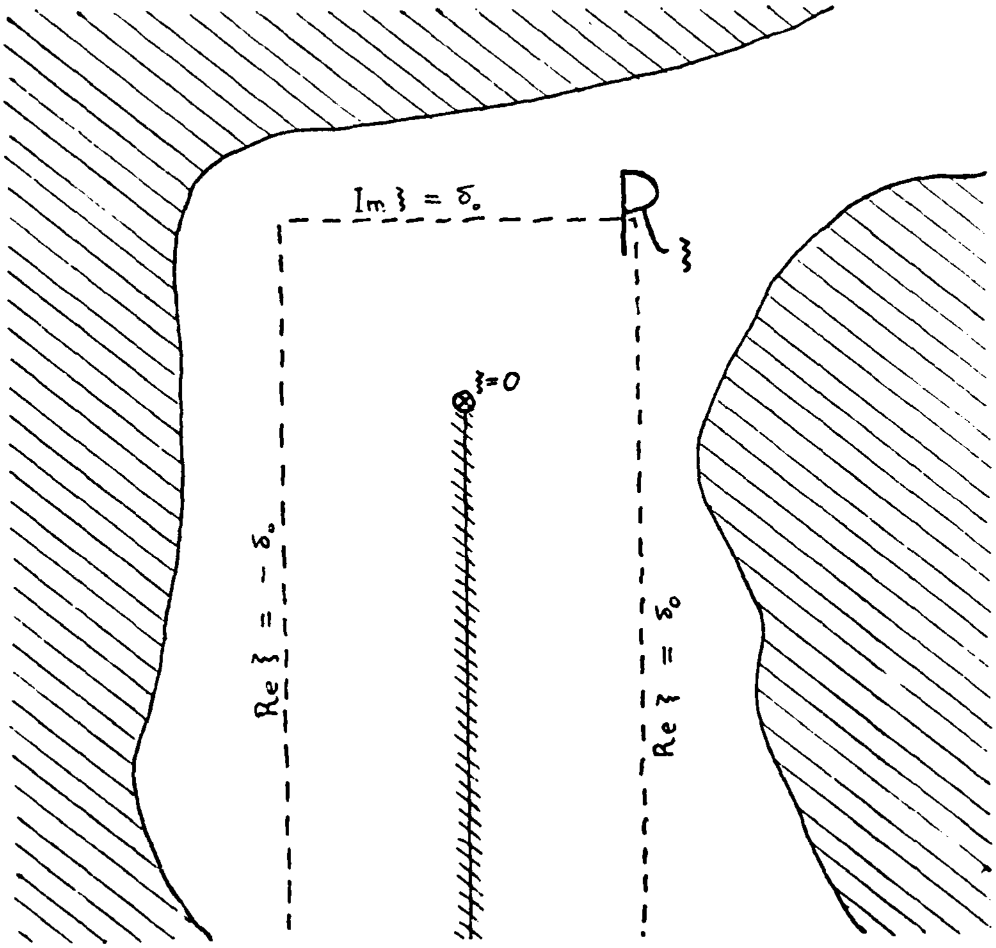
In order to solve (2.5) and (2.6), we shall associate them with integral equations. First, though, we must ensure that the integral equations will exist: we assume that R_ξ is connected and simply connected, and that it contains the set

$$\{\xi : |\mathrm{Re}\xi| \le \delta_o, \mathrm{Im}\xi \le \delta_o\} \setminus \{\xi : \mathrm{Re}\xi = 0,\ 0 > \mathrm{Im}\xi > -\infty\}$$

with $\delta_o > 0$ (Figure 2.1). For example, if $q(z)$ be singular only at $z = 0$, R_ξ could consist of all complex numbers, less $z = 0$ and a branch cut from zero to infinity. We also assume that $\varphi(\xi)$ grows no more than exponentially as $\xi \to -i\infty$ for $-\delta_o \le \mathrm{Re}\xi \le \delta_o$ $(\mathrm{Re}\xi \ne 0)$. That is, we assume that $\varphi(\xi) = \mathcal{O}(\exp(\kappa|\xi|))$ with κ a constant (not depending on ε).

At this point, a choice has been made in regard to the cut (Fig. 2.1), and its implication should be explained. First of all, the original formulation of the Schroedinger equation (1.1) is somewhat arbitrary. It is the most general, linear ordinary differential

FIGURE 2.1 : R_ξ



equation of second order in normal form, but there is a whole family of "Liouville" - transformations $z \to z'(z)$ leaving the normal form of (1.1) invariant, but of course, changing the coefficient $q(z)$ into a different coefficient $Q(z')$ and changing the appearance of a connection problem. This explains why connection problems can come in so many guises and why it is difficult to tell at the outset whether two people are talking about (Liouville transforms of) the same connection problem or about genuinely different ones.

By contrast to z, $\xi(z)$ is the distinguished variable characterizing the oscillatory nature of the solutions. Where an approximation (2.1), (2.2) holds, $|A|$ and $|B|$ represent amplitudes and $-\varepsilon \arg(Aq^{-1/2})$ and $-\varepsilon \arg(Bq^{-1/2)}$, phases; $\xi/2\pi\varepsilon)$ measures distance or time in local wavelengths or periods. Any canonical formulation of the connection problem must therefore by in terms of ξ. The real axis of ξ is the line on which (2.1), (2.2) are purely oscillatory, without exponential growth or decay, and this makes connection between $\xi_\ell < 0$ and $\xi_r > 0$ the most common form of the problem. It leaves a pair of problems, however, one with the cut in the lower half-plane and the other, with the cut in the upper one. These are analogous, dual problems, differing just by an exchange in the roles of $A(\xi)$ and $B(\xi)$. For definiteness, that with the cut in the lower half-plane is selected for study here.

The integral equations

$$B(\xi) - B_\ell = \int_{\xi_\ell}^{\xi} \varphi(s)A(s)e^{\rho s}\,ds \tag{2.8}$$

$$(2.9) \qquad A(\xi) - A_\ell = \int_{\xi_\ell}^{\xi} \varphi(s)B(s)e^{-\rho s}\,ds$$

are immediately associated with (2.6) and (2.5). For the present choice of cut, however, a single integral equation involving A alone, rather than A and B as in (2.9), is more helpful. Towards that end, integrate (2.9) by parts:

$$(2.10) \quad A(\xi) - A_\ell = B(s)\int_{\xi}^{s} \varphi(t)e^{-\rho t}\,dt\Big|_{s=\xi_\ell}^{s=\xi} - \int_{\xi_\ell}^{\xi} \frac{dB(s)}{ds}\,ds \int_{\xi}^{s} \varphi(t)e^{-\rho t}\,dt$$

To clarify the structure of (2.10), let

$$(2.11) \qquad j(\xi)e^{-\rho\xi} = \int_{\xi}^{-i\infty} e^{-\rho t}\varphi(t)\,dt$$

For sufficiently small ε, the exponential factor decreases much faster than φ increases as $t \to -i\infty$, so that this integral (2.11) converges. The kernel in (2.10) is then

$$(2.12) \qquad j(\xi, s)e^{-\rho s} = \int_{s}^{\xi} e^{-\rho t}\varphi(t)\,dt = j(s)e^{-\rho s} - j(\xi)e^{-\rho\xi}$$

and by (2.6), (2.10) becomes

$$(2.13) \quad A(\xi) - A_\ell = B_\ell\, j(\xi, \xi_\ell)\exp(-\rho\xi_\ell) + \int_{\xi_\ell}^{\xi} \varphi(s)A(s)j(\xi, s)\,ds$$

This will be our basic integral equation. We shall solve it for $A(\xi_r)$, returning to the much simpler problem of finding $B(\xi_r)$ only at the end.

We will usually take the path of integration for (2.8) and (2.13) to be an arc of the circle $|s| = \delta$. The radius $\delta = \delta(\varepsilon)$ will be chosen as a compromise between two conflicting needs. First, if q has a singularity or zero at $\xi = 0$, that will determine the connection coefficients, so we must do computations near zero in order to get an answer. On the other hand, if q has a singularity or zero at $\xi = 0$, then φ may be behaved badly enough to make computations excessively difficult if they be done too near $\xi = 0$. Such conflicts will lie at the heart of much of the analysis.

The basic conditions on $\delta(\varepsilon)$ are that:

(2.14) (a) $\delta(\varepsilon) > 0$ (b) $\delta(\varepsilon) \to 0$ (c) $\delta(\varepsilon)/\varepsilon \to \infty$ as $\varepsilon \to 0$

Condition (2.14b) is strengthened in Section 3, equation 3.21.

We take ξ_ℓ and ξ_r to lie on the circle $|\xi| = \delta$; i.e.,

(2.15) $$\xi_\ell = -\delta \qquad \xi_r = \delta$$

The coefficients $A(\xi), B(\xi)$ at $\xi = \xi_\ell$ and $\xi = \xi_r$ are normally approximately equal to the coefficients at greater distances from zero, such as $\xi = -1$ and $\xi = 1$. That will be discussed in Section 6.

3. Close Connection.

3.1. In order to approximate the WKB coefficients on the right, A_r and B_r, in terms of given values of the coefficients on the left, A_ℓ and B_ℓ, we must know something about the singularity between the right and left, at $\xi = 0$. Since we are interested in $q(z)$ which have logarithmic behavior at $z = 0$, we choose an assumption which will cover such q. We assume that:

$$\text{(A)} \qquad \xi\varphi(\xi) \to \gamma \qquad \text{as} \quad |\xi| \to 0$$

uniformly in $\arg \xi$ for which φ is defined; i.e., $\xi \in R_\xi$. Here γ is an arbitrary constant, with $-\infty < \gamma < \frac{1}{2}$.

In Section 5.1, it will be shown that (A) holds with $\gamma = \frac{1}{2}\nu(\nu + 1)^{-1}$ when $q(z) = z^\nu(\log z)^\mu$, with ν and μ real and $\nu > -1$. (A) also covers the "fractional" turning point class studied by Langer, Riekstins and Olver (1977).

To use the basic integral equation (2.13), we must begin by estimating the kernel $j(\xi,s)$, which is complicated by the lack of information on the rate of approach to the limit (A); the full report is postponed to Section 4. The method is to substitute (A) into (2.11) and (2.12). Once the resulting error terms are successfully dealth with, $j(\xi)e^{-\rho\xi}$ will be approximated by an incomplete gamma function:

$$\int_\xi^{-i\infty} e^{-\rho t}(\gamma/t)dt$$

This can be further approximated by a well-known formula, except near $\xi = \xi_\ell$. At $\xi = \xi_\ell$, another method can be used. The most important difficulty in dealing with the error terms involved lies in the fact that $\exp(-\rho t)$ in (2.11) is large when the imagainary part of t does not quickly approach zero; while if t does quickly approach zero, then not only will $\varphi(t)$ in (2.11) grow large, but so will the error term $\varphi(t) - (\gamma/t) = o(t^{-1})$. This sort of problem is handled primarily by care in choosing the different paths of integration on whicn (2.11), (2.12) and their error terms are computed. The results are summarized below. These asymptotic formulae hold uniformiy as $\varepsilon \to 0$, for ξ and s lying on arcs of the circle $|\xi| = |s| = \delta(\varepsilon)$; that is, $|\rho\xi| = |\rho s| = 2\delta/\varepsilon$. These arcs are pictured in Figure 3.1.

(3.1) $\quad j(\xi) = (\gamma + o(1))/(\rho\xi)$ for $0 \le \arg\xi \le \pi - \Theta_o$ (Figure 3.1 (a))

where $\Theta_o > 0$ is an arbitrary constant.

(3.2) $$j(\xi, s)e^{-\rho s} = (\gamma + o(1))[e^{-\rho s}/(\rho s) - e^{-\rho\xi}/(\rho\xi)]$$

for $0 \le \arg\xi \le \pi - \Theta_o$ and $0 \le \arg s \le \pi - \Theta_o$ (Figure 3.1 (a)).

(3.3) $$j(\xi, \xi_\ell)e^{-\rho\xi_\ell} = (\gamma + o(1))[\exp(-\rho\xi_r)/(\rho\xi_r) - \exp(-\rho\xi)/(\rho\xi) - 2\pi i]$$

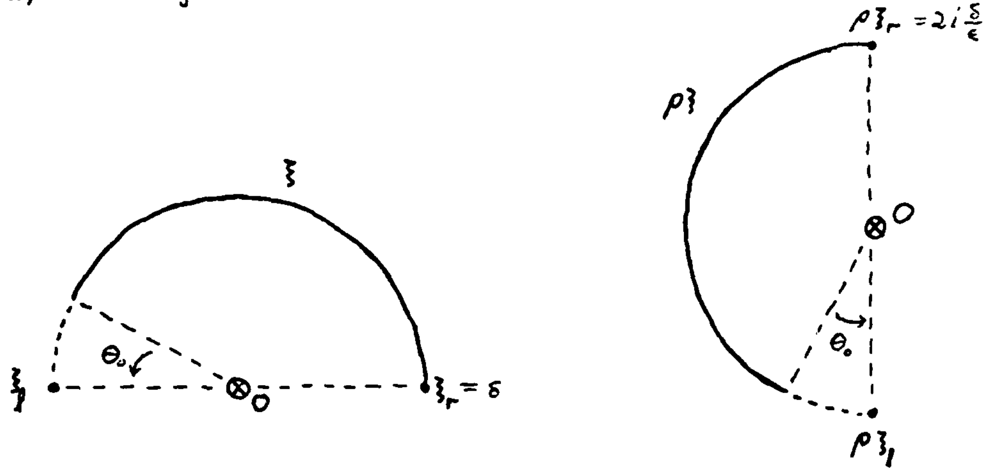
for $0 \le \arg\xi \le \pi - \Theta_o$ (Figure 3.1 (a))

(3.4) $$j(\xi, s)e^{-\rho s} = \Theta(e^{-\rho\xi}/(\rho\xi))$$

for $\frac{\pi}{2} \le \arg\xi \le \arg s \le \pi$ (Figures 3.1 (b) and (c)).

FIGURE 3.1 : Arcs of validity of equations (3.1)-(3.4)

(a) $0 \leq \arg(\zeta) \leq \pi - \theta_0$



(b) $\frac{1}{2}\pi \leq \arg(\zeta) \leq \pi$

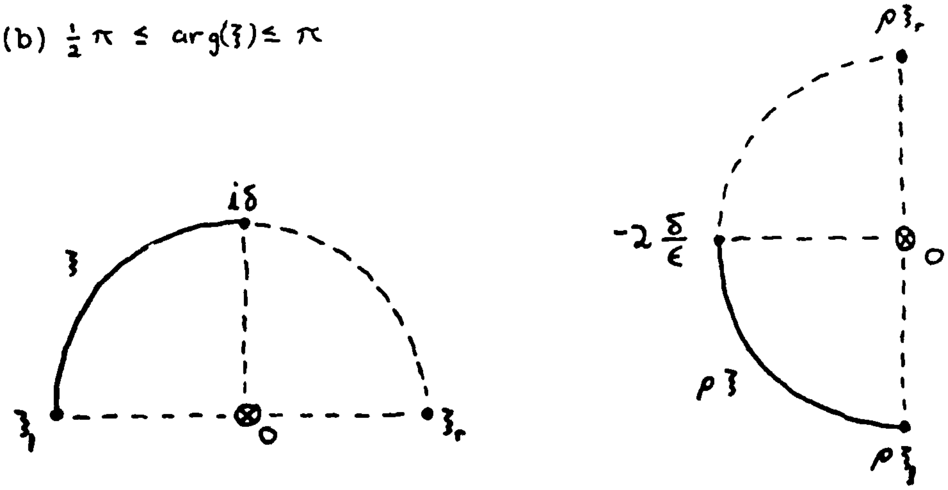
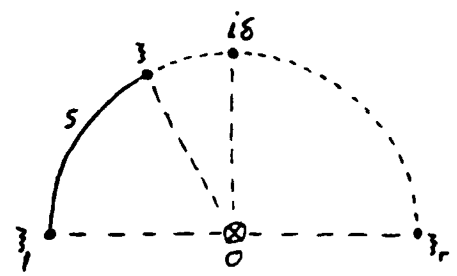
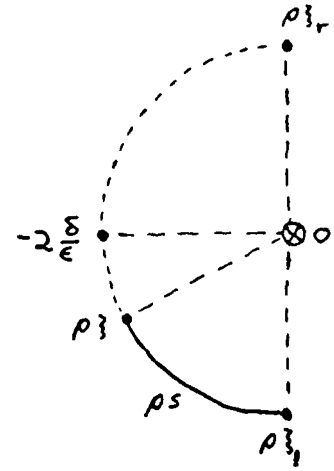


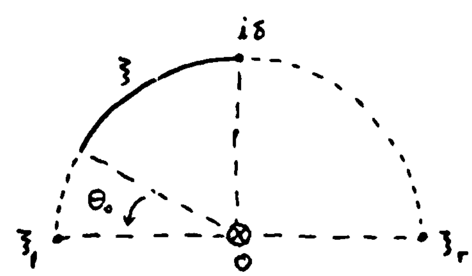
FIGURE 3.1 : Arcs of validity of equations (3.1)-(3.4)

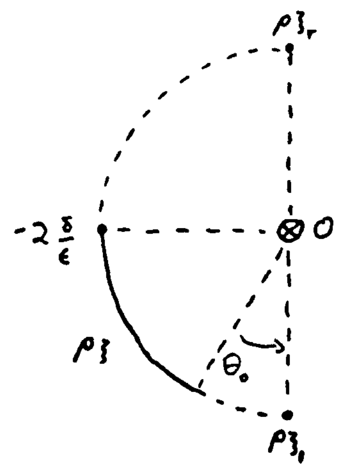
(c) $\frac{1}{2}\pi \le \arg(\zeta) \le \arg(s) \le \pi$





(d) $\frac{1}{2}\pi \le \arg(\zeta) \le \pi - \theta_0$





3.2. We need a bound on $A(\xi)$ before we can compute it. First, note that the derivation of (2.13) can be repeated with ξ_r in place of ξ_ℓ to give

$$(3.5)\quad A(\xi) - A_r = B_r j(\xi, \xi_r) \exp(-\rho\xi_r) + \int_{\xi_r}^{\xi} \varphi(s) A(s) j(\xi, s) ds$$

There is exponential behavior in A; we extract most of it by multiplying $A(\xi)$ by an exponentially small factor:

$$(3.6)\qquad a(\xi) = A(\xi) e^{\rho\xi} .$$

Then equations (2.13) and (3.5) become:

$$(3.7)\quad a(\xi) - a_d \exp(\rho\xi - \rho\xi_d) = B_d j(\xi, \xi_d) \exp(\rho\xi - \rho\xi_d) + \int_{\xi_d}^{\xi} a(s) j(\xi, s) \exp(\rho\xi - \rho s) \varphi(s) ds$$

where "d" can be either "ℓ" or "r". To have the exponential in the integral in (3.7) bounded, we require ξ to lie in the same quadrant as ξ_d (Figure 3.2). That is, if $\xi_d = \xi_\ell$ then $\frac{\pi}{2} \leq \arg\xi \leq \pi$, so that $\pi \leq \arg(\rho\xi) \leq \arg(\rho s) \leq \arg(\rho\xi_\ell) = 3\pi/2$. If $\xi_d = \xi_r$, then $0 \leq \arg\xi \leq \frac{\pi}{2}$, so that $\arg(\rho\xi_r) = \pi/2 \leq \arg(\rho s) \leq \arg(\rho\xi) \leq \pi$.

With such ξ, if s lies between ξ and ξ_d as in the integral of (3.7), then ρs will have greater real part than $\rho\xi$, so that $\exp(\rho\xi - \rho s)$ will be bounded. Moreover, we may use (3.2) or (3.4) to find that, for s in the integral in (3.7) or for $s = \xi_d$,

$$(3.8) \qquad j(\xi, s)\exp(\rho\xi - \rho s) = \mathcal{O}(\rho\xi)^{-1} = \mathcal{O}(\varepsilon/\delta) \to 0$$

Substitute (3.8) and assumption (A) into (3.7).

$$(3.9) \qquad a(\xi) = A_d e^{\rho\xi} + B_d \mathcal{O}(\varepsilon/\delta) + ||a||_\infty^d \, \mathcal{O}\left(\frac{\varepsilon}{\delta} \int_{\xi_d}^{\xi} \frac{ds}{s}\right)$$

where $||a||_\infty^d$ is the sup norm, applied for ξ or s on the quarter-circle allowed (Figure 3.2); that is, $||a||_\infty^\ell = \sup\{|a(s)| : \frac{\pi}{2} \leq \arg(s) \leq \pi, |s| = \delta\}$ and $||a||_\infty^r = \sup\{|a(s)| : 0 \leq \arg(s) \leq \pi/2, |s| = \delta\}$.
Let

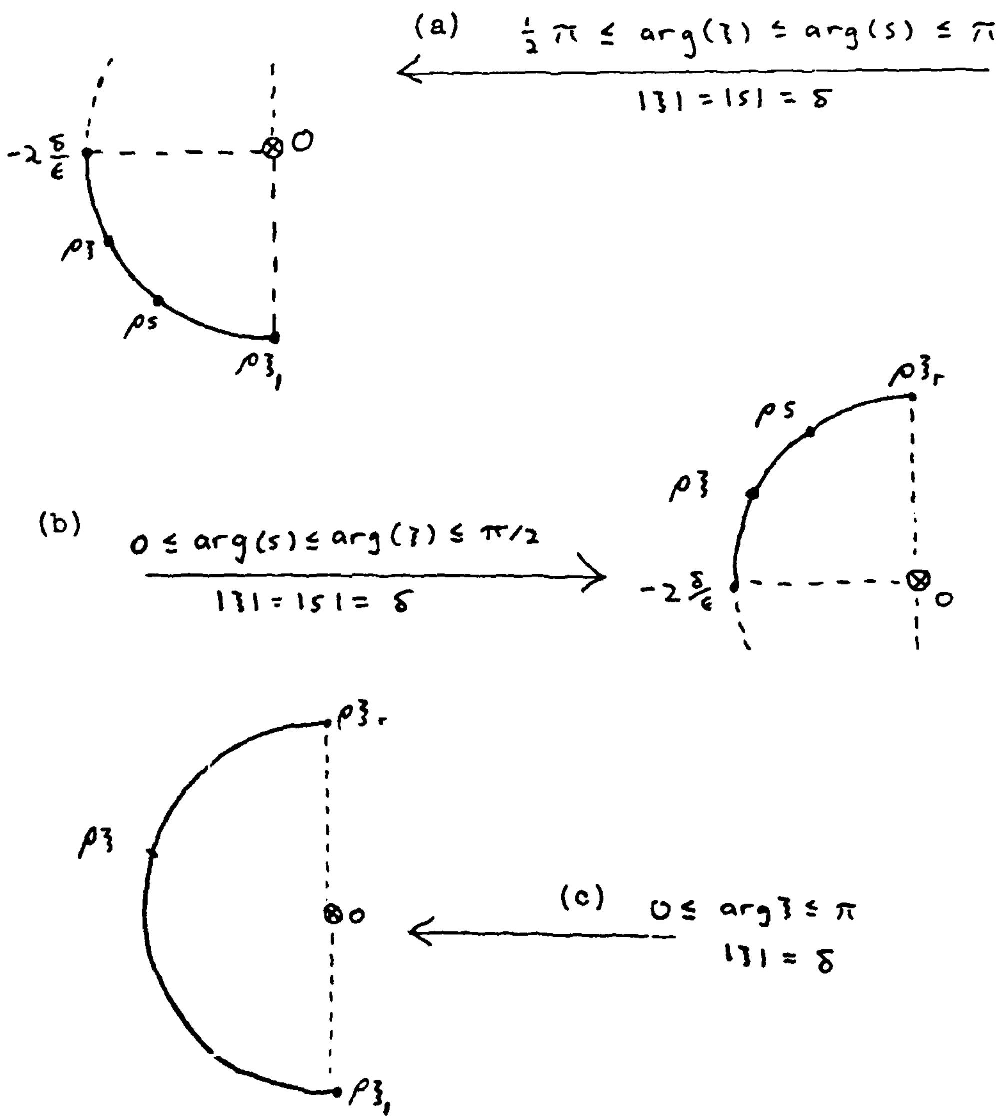
$$(3.10) \qquad C = \max(|A_\ell|, |A_r|, |B_\ell|, |B_r|) .$$

We shall see, of course, that C is usually bounded as $\varepsilon \to 0$, but that is not yet known. The integral in (3.9) is bounded by $\pi/2$, and the exponential decreases (or is constant) in magnitude as $\varepsilon \to 0$, so that (3.9) becomes

$$a(\xi) = \mathcal{O}(C) + \mathcal{O}\left(\frac{\varepsilon}{\delta}\right) ||a||_\infty^d$$

Therefore,

FIGURE 3.2 : Locations of ζ and s in Chapter 3, Section 2



$$||a||_\infty^d = \mathcal{O}(C)$$

and, hence,

(3.11) $$||a||_\infty = \sup \{ |a(\xi)| : |\xi| = \delta, 0 \le \arg \xi \le \pi \} = \mathcal{O}(C)$$

We may use (3.6) to translate this back into the original $A(\xi)$ notation.

(3.12) $$A(\xi) = C\,\mathcal{O}(e^{-\rho\xi}) = C\,\mathcal{O}(e^{2\delta/\varepsilon})$$

for ξ on the entire semicircle $|\xi| = \delta$, $0 \le \arg\xi \le \pi$ (Figure 3.2(c)).

The argument that produced this bound can be carried out more precisely to give a formula for $A(\xi)$ valid on the arc $\frac{1}{2}\pi \le \arg\xi \le \pi - \theta_0$ (Figure 3.1(d)). That is done in Section 5.3.

3.3 Now we can compute $A(\xi)$ by showing it to be close to a known function $A^\gamma(\xi)$. We have described $A(\xi)$ by following three equations:

(2.13) $$A(\xi) = A_\ell + B_\ell\, j(\xi, \xi_\ell) \exp(-\rho\xi_\ell) + \int_{\xi_\ell}^{\xi} A(s) j(\xi, s) \varphi(s) ds$$

$$(2.12) \qquad j(\xi, s)\exp(-\rho s) = \int_s^\xi \exp(-\rho t)\varphi(t)dt$$

$$(A) \qquad \varphi(t) = (\gamma + o(1))/t$$

We define $A^\gamma(\xi)$ by replacing $\varphi(t)$ by <u>exactly</u> γ/t; that is,

$$(3.13) \quad A^\gamma(\xi) = A_\ell + B_\ell\, j^\gamma(\xi, \xi_\ell)\exp(-\rho\xi_\ell) + \int_{\xi_\ell}^{\xi} A^\gamma(s) j^\gamma(\xi, s)\gamma s^{-1} ds$$

$$(3.14) \qquad j^\gamma(\xi, s)\exp(-\rho s) = \int_s^\xi \exp(-\rho t)\gamma t^{-1} dt$$

If $q(z) = z^\nu$, then $\rho(t) = \gamma/t$ with $\gamma = \frac{1}{2}\nu(1+\nu)^{-1}$; thus $A^\gamma(\xi)$ is the modulation coefficient for the special case of $q(z) = z^\nu$. For such q, the solution $w(z)$ to (2.1) can be written explicitly in terms of Hankel functions (Section 5.2), so we know A^γ exactly.

In Section 4.8, we show that j and j^γ are approximately equal in the following sense. The error bounds are defined in terms of

$$(3.15) \qquad h(\varepsilon) = \max(g(\delta), \exp(-2\delta/\varepsilon))$$

where $g(\delta)$ is the maximum error in assumption (A), that is,

$$(3.16) \qquad g(\delta) = \sup\{|\xi\varphi(\xi) - \gamma| : |\xi| \le \delta, \xi \in R_\xi\}$$

Note that $g(\delta) \to 0$ as $\delta \to 0$, so also $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. All of the following 3 equations hold for ξ and s satisfying $|\xi| = |s| = \delta$, and with $\varepsilon \to 0$.

$$(3.17) \qquad j(\xi, s) = (1 + \mathcal{O}(h(\varepsilon)))j^{\gamma}(\xi, s)$$

for $0 \le \arg \xi \le \pi - \theta_o$, $0 \le \arg s \le \pi - \theta_o$ (Figure 3.1 (a))

$$(3.18) \qquad j(\xi, \xi_\ell) = (1 + \mathcal{O}(h))j^{\gamma}(\xi, \xi_\ell)$$

for $0 \le \arg \xi \le \pi - \theta_o$ (Figure 3.1 (a))

$$(3.19) \qquad j(\xi, s) = j^{\gamma}(\xi, s) + \mathcal{O}(h)\,\epsilon\,\delta^{-1}\exp(\rho s - \rho\xi)$$

for $\frac{1}{2}\pi \le \arg \xi \le \arg s \le \pi$ (Figure 3.1 (c)).

Since we want to use this to show that $A(\xi) - A^{\gamma}(\xi)$ is small, we next combine (2.13) and (3.13) into an equation for $A(\xi) - A^{\gamma}(\xi)$.

$$(3.20) \quad A(\xi) - A^{\gamma}(\xi) = B_\ell[j(\xi,\xi_\ell) - j^{\gamma}(\xi,\xi_\ell)]\exp(-\rho\xi_\ell) + $$
$$+ \int_{\xi_\ell}^{\xi} A^{\gamma}(s)[j(\xi, s)\varphi(s) - j^{\gamma}(\xi, s)\gamma s^{-1}]ds +$$
$$+ \int_{\xi_\ell}^{\xi} [A(s) - A^{\gamma}(s)]j(\xi, s)\varphi(s)ds$$

This equation will be solved approximately in the next two subsections. They will depend upon a restriction on the choice of the arbitrary function $\delta(\varepsilon)$. So far we have required that $\varepsilon \ll \delta(\varepsilon) \ll 1$. Now we also require $\delta(\varepsilon)$ to satisfy

(3.21) $$\varepsilon\delta^{-1}g(\delta)\exp(2\delta/\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0$$

Consequently, $h(\varepsilon)$ also satisfies (3.21). Whatever the function $g(\delta)$ may be, this condition (3.21) can be satisfied by some choice of $\delta(\varepsilon)$. For example, we could define a function $\varepsilon(\delta)$ by

(3.22) $$\varepsilon(\delta) = 2\delta\,|\log(g(\delta))|^{-1}$$

This decreases monotonically to zero as $\delta \to 0$ because $g(\delta)$ does; thus $\varepsilon(\delta)$ can be inverted to give $\delta(\varepsilon)$ satisfying both $\varepsilon \ll \delta \ll 1$ and (3.21).

In the case of greatest interest, where $q(z) = z^{\nu}(\log z)^{\mu}$, we have $g(\delta) = \mathcal{O}(\log\delta)^{-1}$, by (5.7), so that the requirements for $\delta(\varepsilon)$ can be satisfied with

$$\delta\varepsilon^{-1} = \frac{1}{2}\log|\log\delta|$$

This increases only slowly; i.e., equation (3.21) has forced us to put $\delta(\varepsilon)$ rather near ε.

3.4. In this section we shall solve (3.20) for ξ in the arc $|\xi| = \delta$, $\frac{1}{2}\pi \leq \arg\xi \leq \pi$ (Figure 3.3). To bound the first bracketed term in (3.20), we use equation (3.19) with $s = \xi_{\ell}$

(3.23) $$[j(\xi,\xi_{\ell}) - j^{\gamma}(\xi,\xi_{\ell})]\exp(-\rho\xi_{\ell}) = \mathcal{O}(h)\,\varepsilon\,\delta^{-1}\exp(-\rho\xi)\,.$$

The integrand in the second term of (3.20) contains A^{γ}, which is bounded by (3.12). For the case $q(z) = z^{\nu}$ which leads to A^{γ}, it is already known, as will be discussed in section 5.2, that equation (2.2) (with A and B constants) holds as close to zero as $|\xi| = \delta$, and that the connection formula (1.4) holds; therefore, C defined by (3.10) satisfies $C = \mathcal{O}(1)$ as $\varepsilon \to 0$, and (3.12) reduces to

$$A^{\gamma}(\xi) = \mathcal{O}(\exp(-\rho\xi)) \tag{3.24}$$

Substitute (A), (3.24), (3.19), and (3.2) into the second term of (3.20).

$$\begin{aligned} \int_{\xi_\ell}^{\xi} & A^{\gamma}(s)[j(\xi, s)\varphi(s) - j^{\gamma}(\xi, s)\gamma s^{-1}]\, ds = \\ &= \int_{\xi_\ell}^{\xi} \mathcal{O}(\exp(-\rho s))[\mathcal{O}(g(\delta))j(\xi, s) + j(\xi,s) - j^{\gamma}(\xi, s)]\gamma s^{-1} ds \\ &= \int_{\xi_\ell}^{\xi} [\mathcal{O}(g)\,\epsilon\,\delta^{-1}e^{-\rho\xi} + \mathcal{O}(h)\,\epsilon\,\delta^{-1}e^{-\rho\xi}]\gamma s^{-1} ds \\ &= \mathcal{O}(h)\,\epsilon\,\delta^{-1}e^{-\rho\xi} . \end{aligned} \tag{3.25}$$

The last term of (3.20) can be estimated by the same method as in section 2, again using equation (3.8). Define

$$b(\xi) = [A(\xi) - A^{\gamma}(\xi)] \exp(\rho\xi) \tag{3.26}$$

then (3.20), (3.23), and (3.25) may be combined to give

$$(3.27)\quad b(\xi) = \mathcal{O}(h)\,\epsilon\,\delta^{-1} + \int_{\xi_\ell}^{\xi} b(s)\exp(\rho\xi - \rho s)j(\xi,s)\varphi(s)ds$$

Now substitute (3.8), and use the norm definition after (3.9) and the fact that $\int \varphi(s)ds = \mathcal{O}(1)\int s^{-1}ds = \mathcal{O}(1)$.

$$(3.28)\quad b(\xi) = \mathcal{O}(h)\,\epsilon\,\delta^{-1} + ||b||_{\infty}^{\ell}\,\mathcal{O}(\varepsilon\delta^{-1})$$

It follows that

$$(3.29)\quad b(\xi) = \mathcal{O}(h)\,\epsilon/\delta$$

By (3.26), this may be translated back into our original terminology. For the ξ allowed in this section (Figure 3.3),

$$(3.30)\quad \begin{aligned} A(\xi) - A^{Y}(\xi) &= \mathcal{O}(h)\,\epsilon\,\delta^{-1}\exp(-\rho\xi) \\ &= \mathcal{O}(h)\,\epsilon\,\delta^{-1}\exp(2\delta/\varepsilon) \end{aligned}$$

Now equation (3.21) gives us the result, that $A^{Y}(\xi)$ is a first approximation to $A(\xi)$. For ξ in the upper-left quarter-circle (Figure 3.3):

$$(3.31)\quad A(\xi) = A^{Y}(\xi) + o(1)$$

3.5. In this section we shall solve (3.20) for ξ in the arc $|\xi| = \delta$, $0 \le \arg\xi \le \frac{1}{2}\pi$ (Figure 3.3). That will give us the solution to the connection problem. The first bracketed term in (3.20) may be bounded by means of equations (3.18), (3.3), and (3.21).

$$\begin{aligned} [j(\xi, \xi_\ell) - j^\gamma(\xi, \xi_\ell)] \exp(-\rho\xi_\ell) &= \mathcal{O}(h) j(\xi, \xi_\ell) \exp(-\rho\xi_\ell) \\ &= \mathcal{O}(h)[\mathcal{O}(1) + \mathcal{O}(\rho\xi)^{-1} \exp(-\rho\xi)] \\ &= \mathcal{O}(h)[\mathcal{O}(1) + \mathcal{O}(\varepsilon\delta^{-1} \exp(2\delta/\varepsilon))] \\ &= o(1) \end{aligned} \tag{3.32}$$

For the second term of (3.20) we start by substituting assumption (A) and equations (3.16) and (3.24)

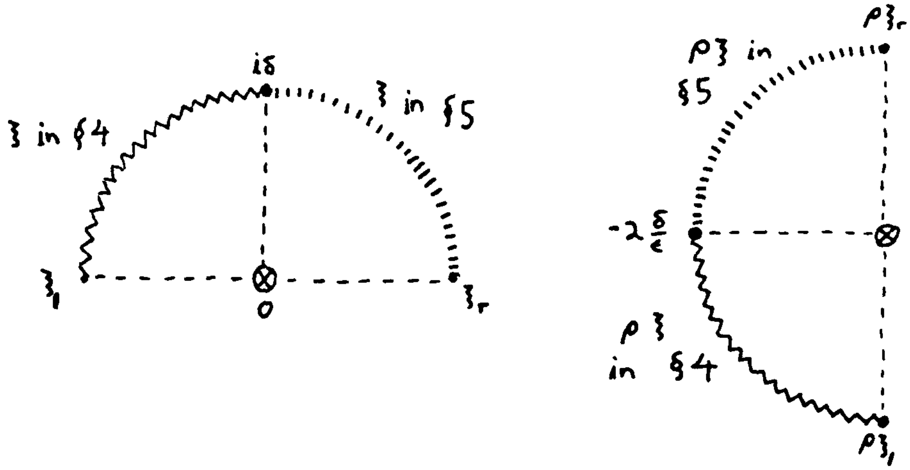
$$\begin{aligned} &\int_{\xi_\ell}^{\xi} A^\gamma(s)[j(\xi, s)\varphi(s) - j^\gamma(\xi, s)\gamma s^{-1}]\, ds \\ &= \int_{\xi_\ell}^{\xi} \mathcal{O}(e^{-\rho s})[\mathcal{O}(g) j(\xi, s) + j(\xi, s) - j^\gamma(\xi, s)]\, \gamma s^{-1}\, ds \end{aligned} \tag{3.33}$$

By (3.2) and (3.4), we can take care of the first term in brackets:

$$\begin{aligned} j(\xi, s) e^{-\rho s} &= \mathcal{O}((\rho\xi)^{-1} e^{-\rho\xi}) + \mathcal{O}((\rho s)^{-1} e^{-\rho s}) \\ &= \mathcal{O}(\varepsilon\delta^{-1} \exp(2\delta/\varepsilon)) \end{aligned} \tag{3.34}$$

For the rest of the bracketed expression, we deal separately with s in each of the two quadrants in Figure 3.3.

FIGURE 3.3 : Location of ξ for which (3.7) is solved in sections 4 and 5



If s be in the right quadrant, $0 \leq \arg s \leq \frac{1}{2}\pi$, then by (3.17) and (3.2)

$$[j(\xi, s) - j^{\gamma}(\xi, s)]e^{-\rho s} = \mathcal{O}(h)j(\xi, s)e^{-\rho s} = \mathcal{O}(h)(\rho s)^{-1}e^{-\rho s}$$

(3.35)
$$= \mathcal{O}(h)\,\epsilon\,\delta^{-1}\exp(2\delta/\varepsilon)$$

On the other hand, if s be in the left quadrant, $\frac{1}{2}\pi \leq s \leq \pi$, then we must first split up $j(\xi, s)$, before applying equations (3.18), (3.3), and (3.19).

$$[j(\xi, s) - j^{\gamma}(\xi, s)]e^{-\rho s} = [j(\xi,\xi_{\ell})\exp(-\rho\xi_{\ell}) - j(s,\xi_{\ell})\exp(-\rho\xi_{\ell})]$$

$$- [j^{\gamma}(\xi,\xi_{\ell})\exp(-\rho\xi_{\ell}) - j^{\gamma}(s,\xi_{\ell})\exp(-\rho\xi_{\ell})]$$

(3.36)
$$= \mathcal{O}(h)j(\xi,\xi_{\ell})\exp(-\rho\xi_{\ell}) - \mathcal{O}(h)\,\epsilon\,\delta^{-1}\exp(-\rho s)$$

$$= \mathcal{O}(h)[\varepsilon\,\delta^{-1}\exp(-\rho\xi) + \mathcal{O}(1)] - \mathcal{O}(h)\,\epsilon\,\delta^{-1}\exp(-\rho s)$$

$$= \mathcal{O}(h)[\epsilon\,\delta^{-1}\exp(2\delta/\varepsilon) + \mathcal{O}(1)]$$

Now we can put together equations (3.34), (3.35) and (3.36), by substituting them into (3.33). Then use (3.21).

$$\int_{\xi_{\ell}}^{\xi} A^{\gamma}(s)[j(\xi, s)\varphi(s) - j^{\gamma}(\xi, s)\gamma s^{-1}]\,ds$$

$$= \int_{\xi_{\ell}}^{\xi} \mathcal{O}(h)\,\epsilon\,\delta^{-1}\exp(2\delta/\varepsilon)s^{-1}\,ds$$

$$= o(1)$$

Next break up the last integral in (3.17) into two parts, so that when (3.32) and (3.36) are substituted, it becomes:

$$(3.37)\quad [A(\xi)- A^{\gamma}(\xi)] = o(1) + \int_{\xi_\ell}^{1\delta} [A(s) - A^{\gamma}(s)]j(\xi, s)\varphi(s)\,ds + \int_{1\delta}^{\xi}[A(s) - A^{\gamma}(s)]\,j(\xi, s)\varphi(s)\,ds$$

The first integral in (3.37) can be handled with (3.2) and (3.30), the result of the previous section.

$$(3.38)\quad \begin{aligned} \int_{\xi_\ell}^{1\delta} [A(s) - A^{\gamma}(s)]j(\xi, s)\varphi(s)ds &= \int_{\xi_\ell}^{1\delta} \Theta(h(\varepsilon))(\varepsilon/\delta)e^{-\rho s}j(\xi,s)\varphi(s)\,ds \\ &= \Theta(h(\varepsilon))(\varepsilon/\delta) \int_{\xi_\ell}^{1\delta} (\gamma + o(1))^2((\rho s)^{-1}e^{-\rho s} - (\rho\xi)^{-1}e^{-\rho\xi})s^{-1}ds \\ &= \Theta(h(\varepsilon))(\varepsilon/\delta)^2 \exp(2\delta/\varepsilon) \\ &= o(\varepsilon/\delta) \end{aligned}$$

In the last integral in (3.37) we may estimate $j(\xi,s)$ by (3.2) and the fact that $\exp(-\rho\xi)$ is smaller than $\exp(-\rho s)$ there, and we may estimate $\varphi(s)$ by (A). The result is that (3.37) and (3.38) imply

$$(3.39)\quad [A(\xi) - A^{\gamma}(\xi)] = o(1) + \Theta(\varepsilon/\delta)\,||A(s) - A^{\gamma}(s)||_{\infty}$$

It follows that in the quadrant studied in this section, and hence in the entire semicircle $|\xi| = \delta,\ 0 \le \arg\xi \le \pi$ (Figure 3.3),

$$A(\xi) = A^{\gamma}(\xi) + o(1) \ . \tag{3.40}$$

3.6. The approximation (3.40) for $A(\xi)$ gives us the connection formula. It will be shown in Section 5 that

$$A^{\gamma}(\xi_r) = A_{\ell} - 2B_{\ell}\, i \sin(\gamma\pi) + o(1) \qquad (\text{for } -\infty < \gamma < \tfrac{1}{2}) \tag{3.41}$$

Therefore, by (3.40)

$$A_r = A_{\ell} - 2B_{\ell}\, i \sin(\gamma\pi) + o(1) \tag{3.42}$$

Of course, A_r and A_{ℓ} are the values of the modulation coefficient $A(\xi)$ at $\xi = \delta(\varepsilon), -\delta(\varepsilon)$, even though we may really be interested in $A(\xi)$ for larger ξ, e.g., $\xi = 1, -1$. However, in Chapter 6 it will be shown that, given a very slight strengthening of assumption (A), $A_r = A(1) + o(1)$, $A_{\ell} = A(-1) + o(1)$, and $B_{\ell} = B(-1) + o(1)$; thus the same connection formula (3.42) will still hold with $A_r = A(1)$ and $A_{\ell} = A(-1)$.

Equation (3.42) solves only half of the connection problem; the other half is to show that $B_r = B_{\ell} + o(1)$. Given (A''), the strengthening of A in Chapter 6 $B_r = B(1) + o(1)$ and $B_r = B(-1) + o(1)$. Thus it suffices to show $B(1) = B(-1) + o(1)$. This follows from the well-known WKB formula (2.2) and the fact that

$B(\xi)$ is the coefficient of the dominant term of (2.4), for positive imaginary ξ.

More precisely, we apply (2.2) twice to $\xi = i$, once in a region R_L containing $\xi = -1$ and once in a region R_R containing $\xi = 1$ (Figure 3.4).

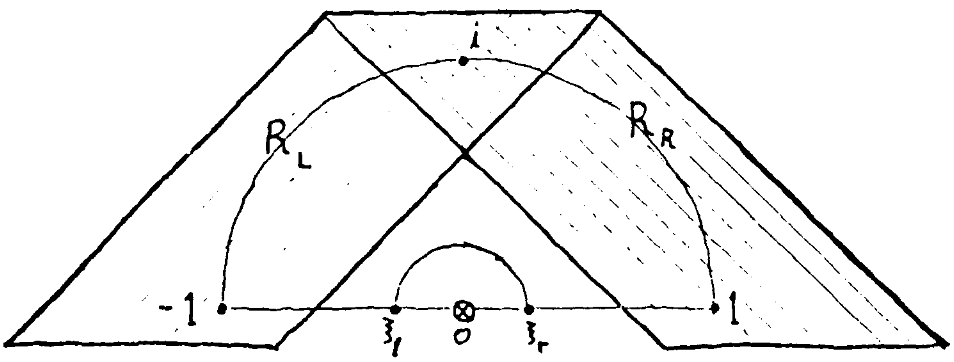
The resulting equations are

$$\begin{aligned} q^{1/2}w\Big|_{z=z(i)} &= [A(-1) + o(1)]e^{-1/\varepsilon} + [B(-1) + o(1)]e^{1/\varepsilon} \\ &= [B(-1) + o(1)]e^{1/\varepsilon} \\ q^{1/2}w\Big|_{z=z(i)} &= [A(1) + o(1)]e^{-1/\varepsilon} + [B(1) + o(1)]e^{1/\varepsilon} \\ &= [B(1) + o(1)]e^{1/\varepsilon} \end{aligned}$$

Therefore,

$$B(1) = B(-1) + o(1) \tag{3.43}$$

FIGURE 3.4 : Connecting B(-1) to B(1)



4. Properties and Error Bounds of the Kernel.

4.1 The computations in Chapter 3 depended upon equations (3.2)-(3.4) and (3.15) for $j(\xi,s)$. In this section we shall derive those estimates, using assumption (A) of Section 3, but not the restriction (3.18) on $\delta(\varepsilon)$. The first step is to compute

$$j(\xi)e^{-\rho\xi} = \int_{\xi}^{-i\infty} e^{-\rho s}\varphi(s)\,ds \tag{4.1}$$

Assumption (A) suggests that we approximate $\varphi(s)$ in (4.1) by γ/s, but in order to keep the error term small we must keep the increasing exponential factor $\exp(-\rho s)$ under control, and that requires care in choosing the path of integration and computing error bounds.

For the computation of $j(\xi)$, the path of integration will be that shown in Figure 4.1. We have ξ on the circle between $\xi_\ell = -\delta$ and $\xi_r = \delta$, not too close to ξ_ℓ; i.e., $|\xi| = \delta$ and $0 \le \arg\xi \le \pi - \theta_o$, with $\theta_o > 0$. We build the path out of four curves. The first is Λ_1, a vertical line drawn from $\xi = \xi_R + i\xi_1$ to $\hat{\xi}$, the point with the same real part and a lower imaginary part, $\hat{\xi}_1 = \min(\frac{1}{2}\xi_1, \xi_1 - \frac{1}{2}\xi_R)$. This is chosen so that $\mathrm{Im}(s)$, and hence $|\exp(-\rho s)|$, will be decreasing along Λ_1. The second curve is Λ_2, a horizontal line from $\hat{\xi}$ to $\tilde{\xi}$, the point with the same imaginary part and whichever nonegative real part makes $\tilde{\xi}$ lie on the circle.

That is, $\tilde{\xi}_i = \hat{\xi}_i$, $\tilde{\xi}_R \geq 0$ and $|\tilde{\xi}| = \delta$. This curve is chosen to avoid $s = 0$, where $\varphi(s)$ becomes large, without resorting to an increase in $|\exp(-\rho s)|$. Then most of the magnitude of the integral in (4.1) will come from near ξ, where $\exp(-\rho s)$ is biggest.

Finally, the third curve Λ_3 is the arc from $\tilde{\xi}$ to $\xi_b = \delta \exp(-\pi i/4) = \delta\sqrt{2}/2 - i\delta\sqrt{2}/2$ and the fourth curve Λ_4 is the vertical ray straight down to $-i\infty$. These curves are chosen to lead to a practically fixed point, with $|\exp(-\rho s)|$ monotonic. On Λ_3 and Λ_4, $|\exp(-\rho s)|$ decreases to zero exponentially faster in s than $|\varphi(s)|$ can increase.

Now we shall approximate $\varphi(s)$ by γ/s and split the remaining error term into four parts, one for each of the four curves Λ_1, Λ_2, Λ_3, and Λ_4. By (4.1)

$$j(\xi)e^{-\rho\xi} = \gamma \int_{\xi}^{-i\infty} e^{-\rho s} s^{-1} ds + \sum_{i=1}^{4} \int_{\Lambda_i} e^{-\rho s}(s\varphi(s) - \gamma)s^{-1} ds .$$

Now change variables in the first integral and pull $g(\delta)$ out of the next three integrals

$$(4.2) \qquad j(\xi)e^{-\rho\xi} = \gamma \int_{\rho\xi}^{\infty} e^{-u} u^{-1} du$$

$$+ \mathcal{O}(g(\delta)) \sum_{i=1}^{3} \int_{\Lambda_i} |e^{-\rho s} s^{-1} ds|$$

$$+ \mathcal{O}(1) \int_{\Lambda_4} |\exp(-\rho s)(\varphi(s) - \gamma/s) ds| .$$

This holds as $\varepsilon \to 0$, uniformly for ξ on the upper semicircle, $|\xi| = \delta$.

The first term in (4.2) is the incomplete gamma function, $\Gamma(0,\rho\xi)$. We may approximate this by [Olver, 1974, page 110]:

$$(4.3) \qquad \Gamma(0,\rho\xi) = \int_{\rho\xi}^{\infty} e^{-u}u^{-1}du \sim (\rho\xi)^{-1}\exp(-\rho\xi) \quad \text{as} \quad \varepsilon \to 0$$

uniformly for ξ on the upper semicircle, not too close to ξ_i; i.e., $|\xi| = \delta$ and $0 \le \arg\xi \le \pi - \theta_0$ (Figure 3.1 (a)).

Once we show the other four terms of (4.2) to be smaller, this will give us a simple formula for $j(\xi)$.

4.2. The first of the four error terms in (4.2) may be estimated by writing it in terms of real variables and integrating by parts to isolate the contribution of the path segment near ξ, where the integrand is biggest.

$$(4.4) \quad \int_{\Lambda_1} |e^{-\rho s}s^{-1}ds| = \int_0^{\xi_i-\hat{\xi}_i} \exp(-i\rho(\xi_i - t))|\xi - it|^{-1}dt$$

$$= \exp(-i\rho\xi_i)[|\rho\xi|^{-1} - |\rho\xi - i\rho(\xi_i - \hat{\xi}_i)|^{-1}\exp(i\rho(\xi_i - \hat{\xi}_i))$$

$$+ |\rho|^{-1}\int_0^{\xi_i-\hat{\xi}_i} \exp(i\rho t)(\xi_i - t)|\xi - it|^{-3}dt]$$

The first term in the brackets is $\frac{1}{2}\varepsilon/\delta$. The exponential in the second term is bounded because $i\rho < 0$ and $\xi_i - \hat{\xi}_i > 0$. The denominator of the second term is $|\rho\hat{\xi}|$.

For $\xi_{\Re} \leq \xi_i$, we have $\hat{\xi} = \xi_{\Re} + \frac{1}{2} i\xi_i$, so that $|\hat{\xi}| \geq \frac{1}{2}\xi_i$. Moreover, for such ξ, $\pi/4 \leq \arg\xi \leq \pi - \theta_o$, so that

$$(4.5) \qquad \xi_i/|\xi| \geq \sin\theta_o > 0 \ .$$

Thus, $|\hat{\xi}| \geq \frac{1}{2}|\xi| \sin\theta_o = \frac{1}{2}\delta \sin\theta_o$.

On the other hand, if $\xi_{\Re} \geq \xi_i$, then $0 \leq \arg\xi \leq \pi/4$, so -

$$(4.6) \qquad \xi_{\Re}/|\xi| \geq \cos(\pi/4) = 1/\sqrt{2} \ .$$

Therefore, $|\hat{\xi}| \geq |\xi_{\Re}| \geq |\xi|/\sqrt{2} > |\xi| (\sin\theta_o)/2 = \frac{1}{2}\delta \sin\theta_o$.

In either case, the second term in the brackets in (4.4) is

$$(4.7) \qquad - |\rho\hat{\xi}|^{-1}\exp(i\rho(\xi_i - \hat{\xi}_i)) = |\rho|^{-1}|\hat{\xi}|^{-1}\,\mathcal{O}(1)$$

$$\leq (\frac{\varepsilon}{2})(\frac{1}{2}\delta\sin\theta_o)^{-1}\,\mathcal{O}(1) = \mathcal{O}(\varepsilon/\delta) \ .$$

For the third term in the brackets of (4.4), we examine the quotient in the integrand, $(\xi_i - t)|\xi - it|^{-3}$. If $\xi_{\Re} \leq \xi_i$, then

$$|\xi - it| \geq |\mathrm{Im}(\xi - it)| \geq \hat{\xi}_i = \frac{1}{2}\xi_i \ .$$

Therefore, and by (4.5),

$$(4.8)\quad (\xi_i - t)|\xi - it|^{-3} \leq \xi_i(\tfrac{1}{2}\xi_i)^{-3} = 8\xi_i^{-2} = \Theta(|\xi|^{-2}) = \Theta(\delta^{-2}) .$$

On the other hand, if $\xi_\ell \geq \xi_i$, then we may use (4.6).

$$(4.9)\quad (\xi_i - t)|\xi - it|^{-3} \leq \xi_i \xi_\ell^{-3} = \Theta(\xi_i|\xi|^{-3}) = \Theta(\delta^{-2}) .$$

Substituting these bounds and the definition of $\rho = 2i/\varepsilon$ into the last term of (4.4) shows that

$$(4.10)\quad |\rho|^{-1} \int_0^{\xi_i - \hat{\xi}_i} \exp(i\rho t)(\xi_i - t)|\xi - it|^{-3} dt$$

$$= \Theta(\varepsilon\delta^{-2}) \int_0^{\xi_i - \hat{\xi}_i} \exp(i\rho t) dt = \Theta(\varepsilon/\delta)^2 [1 - \exp(i\rho(\xi_i - \hat{\xi}_i))]$$

$$= \Theta(\varepsilon/\delta)^2 .$$

This completes an estimate of (4.4):

$$(4.11)\quad \int_{\Lambda_1} |e^{-\rho s} s^{-1} ds| = \exp(-i\rho\xi_i)[\Theta(\varepsilon/\delta) + \Theta(\varepsilon/\delta) + \Theta(\varepsilon/\delta)^2]$$

$$= \Theta((\rho\xi)^{-1} \exp(-\rho\xi)) .$$

and the first of the four error terms in (4.2) is

$$(4.12)\quad \Theta(g(\delta)) \int_{\Lambda_1} |\exp(-\rho s) s^{-1} ds| = \Theta(g(\delta))(\rho\xi)^{-1} \exp(-\rho\xi)$$

$$= o(\rho\xi)^{-1} \exp(-\rho\xi) .$$

This summary error bound is good enough for our purposes because the first term of (4.2) is, by (4.3), approximately $\gamma(\rho\xi)^{-1}\exp(-\rho\xi)$. It takes care of the biggest of the error terms in (4.2), because the exponential factor in the integrant is biggest near ξ.

4.3. The second of the four error terms in (4.2) can be integrated explicitly as follows.

$$\int_{\Lambda_2} |\exp(-\rho s)s^{-1}ds| = \exp(-i\rho\hat{\xi}_i)|\int_{\xi_R}^{\tilde{\xi}_R} |t + i\hat{\xi}_i|^{-1}dt|$$

$$(4.13) \quad = \exp(-i\rho\hat{\xi}_i)|\log(t + |t + i\hat{\xi}_i|)|_{t=\xi_R}^{t=\tilde{\xi}_R}|$$

$$= \exp(-i\rho\hat{\xi}_i)|\log[(\xi_R + |\hat{\xi}|)^{-1}(\tilde{\xi}_R + |\tilde{\xi}|)]|$$

To get a bound for (4.13), we look first at the denominator in the logarithm. For $\xi_R \leq 0$, $\hat{\xi} = \xi_R + i\frac{1}{2}\xi_i$, so that

$$\hat{\xi}_i/|\hat{\xi}| = \frac{1}{2}\xi_i/|\xi| \geq \frac{1}{2}\sin\theta_o$$

and

$$(4.14) \quad \xi_R + |\hat{\xi}| = |\hat{\xi}| - |\xi_R| = (|\hat{\xi}| + |\xi_R|)^{-1}(|\hat{\xi}|^2 - |\xi_R|^2)$$

$$\geq \frac{1}{2}|\hat{\xi}|^{-1}|\hat{\xi}_i|^2 \geq |\hat{\xi}|(\sin\theta_o)^2/8$$

This bound also holds if $\xi_{\mathcal{R}} \geq 0$, for then

$$(4.15) \qquad \xi_{\mathcal{R}} + |\hat{\xi}| \geq |\hat{\xi}| > |\hat{\xi}| (\sin \theta_o)^2/8 \ .$$

These two inequalities, together with (4.5) and (4.6), can give us an upper bound for the quotient in (4.13):

$$(4.16) \quad (\xi_{\mathcal{R}} + |\hat{\xi}|)^{-1} (\tilde{\xi}_{\mathcal{R}} + |\tilde{\xi}|) \leq (|\hat{\xi}| (\sin \theta_o)^2/8)^{-1} (2|\tilde{\xi}|)$$

$$= |\hat{\xi}|^{-1} (\sin \theta_o)^{-2}\, 16\,\delta \leq (\tfrac{1}{2}\delta \sin \theta_o)^{-1} (\sin \theta_o)^{-2}\, 16\delta$$

$$= 32 \sin^{-3} \theta_o \ .$$

We also need a lower bound for this quotient. First, if $\xi_{\mathcal{R}} \leq \xi_i$, then $\hat{\xi} = \xi_{\mathcal{R}} + \frac{1}{2} i\xi_i$, so that

$$|\hat{\xi}| \leq |\xi| = \delta = |\tilde{\xi}| \ .$$

But $\tilde{\xi} = \tilde{\xi}_{\mathcal{R}} + i\hat{\xi}_i$; hence, $\tilde{\xi}_{\mathcal{R}} \geq \xi_{\mathcal{R}}$ (Figure 4.1 (a)). Therefore,

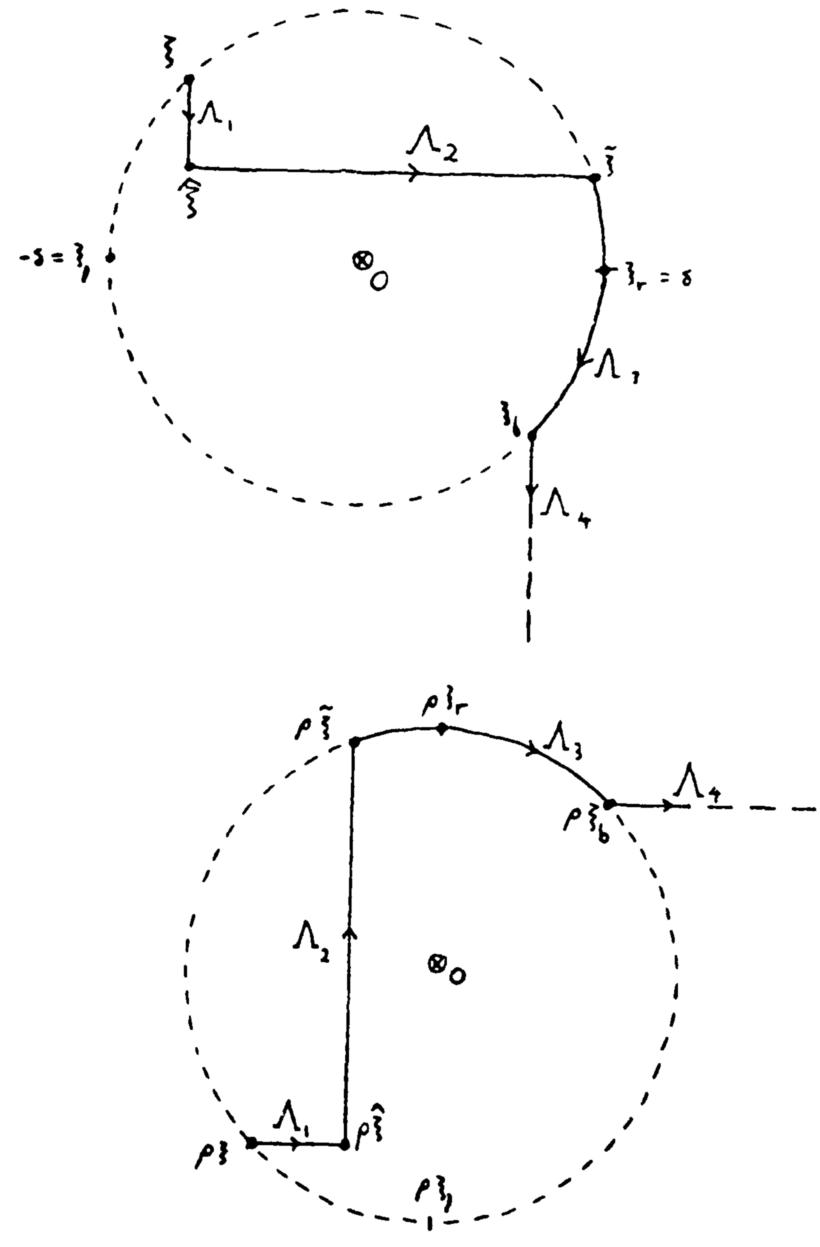
$$(4.17) \quad (\xi_{\mathcal{R}} + |\hat{\xi}|)^{-1} (\tilde{\xi}_{\mathcal{R}} + |\tilde{\xi}|) \geq (\xi_{\mathcal{R}} + |\hat{\xi}|)^{-1} (\xi_{\mathcal{R}} + |\hat{\xi}|) = 1 > (\sin \theta_o)^3/32 \ .$$

On the other hand, if $\xi_{\mathcal{R}} \geq \xi_i$, then

$$|\hat{\xi}_i| = |\xi_i - \tfrac{1}{2}\xi_{\mathcal{R}}| \leq \max(\xi_i, \tfrac{1}{2}\xi_{\mathcal{R}}) \leq \delta \ .$$

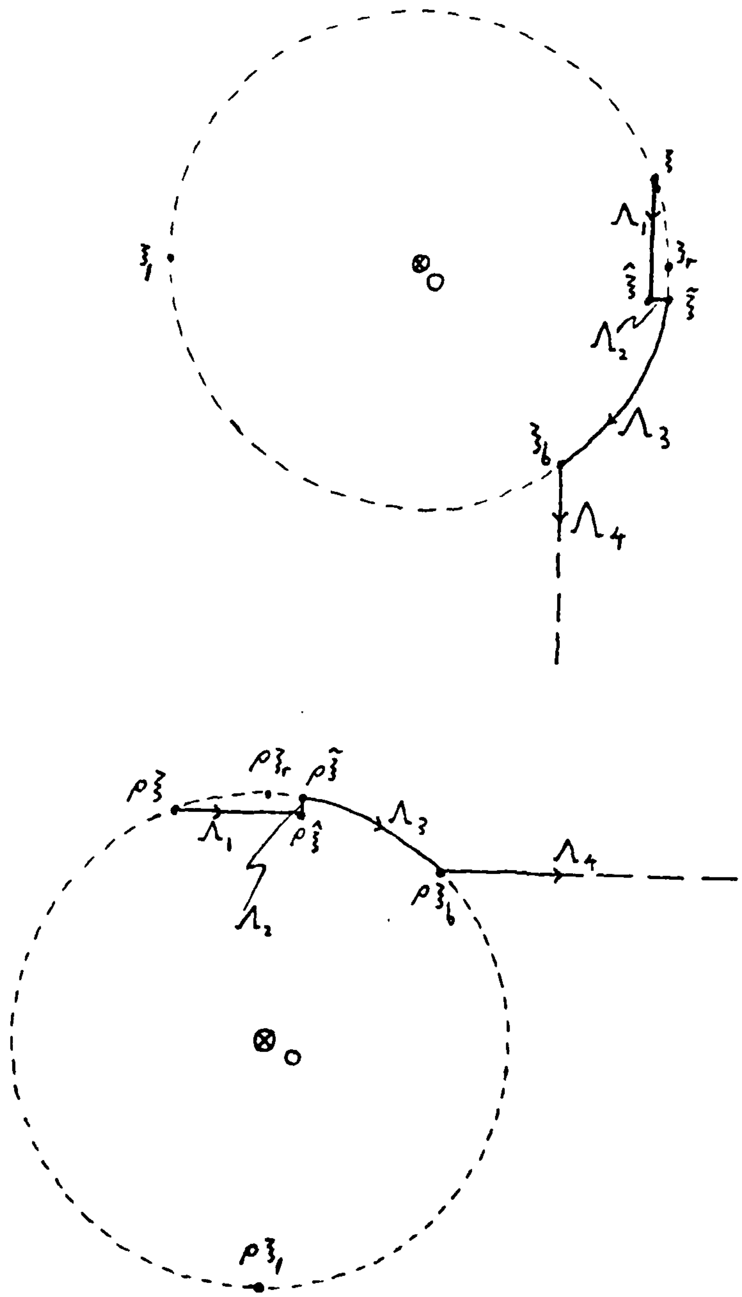
FIGURE A.1 : Paths of integration for $J(\xi)$

(a) $\xi_R \le \xi_i$



<u>FIGURE A.1</u> : Paths of integration for $j(\xi)$

(b) $\xi_R \geq \xi_i$



Consequently,

$$|\hat{\xi}| \le \xi_{\mathcal{R}} + |\hat{\xi}_i| < 2\delta \ .$$

Therefore,

$$(4.20)\quad (\xi_{\mathcal{R}} + |\hat{\xi}|)^{-1}(\tilde{\xi}_{\mathcal{R}} + |\tilde{\xi}|) \ge (\delta + 2\delta)^{-1}|\tilde{\xi}| = 1/3 > (\sin\theta_o)^3/32 \ .$$

Now that we have both an upper bound and a lower bound for this quotient, we can estimate its logarithm.

$$(4.19)\quad |\log[(\xi_{\mathcal{R}} + |\hat{\xi}|)^{-1}(\tilde{\xi}_{\mathcal{R}} + |\tilde{\xi}|)]|$$

$$= \max\{\log[(\xi_{\mathcal{R}} + |\hat{\xi}|)^{-1}(\tilde{\xi}_{\mathcal{R}} + |\tilde{\xi}|)], \log[(\xi_{\mathcal{R}} + |\hat{\xi}|)(\tilde{\xi}_{\mathcal{R}} + |\tilde{\xi}|)^{-1}]\}$$

$$< \log[32(\sin\theta_o)^{-3}]$$

From (4.13), therefore

$$(4.20)\quad \int_{\Lambda_2} |\exp(-\rho s)s^{-1}ds| < \exp(-i\rho\hat{\xi}_i)\log(32\sin^{-3}\theta_o) \ .$$

But $\hat{\xi}_i$ is bounded below ξ_i. For, if $\xi_{\mathcal{R}} \le \xi_i$, then by (4.5)

$$\xi_i - \hat{\xi}_i = \frac{1}{2}\xi_i \ge \frac{1}{2}|\xi|\sin\theta_o = \frac{1}{2}\delta\sin\theta_o \ .$$

If $\xi_R \geq \xi_i$ then by (4.6),

$$\xi_i - \hat{\xi}_i = \frac{1}{2}\xi_R \geq \frac{1}{2}|\xi|/\sqrt{2} = \delta\sqrt{2}/4 .$$

Therefore, (4.20) gives us room to keep the integral smaller than an incomplete gamma function:

$$(4.21) \quad \int_{\Lambda_2} |\exp(-\rho s)s^{-1}ds| = \mathcal{O}(\exp(-i\rho\hat{\xi}_i))$$

$$= \mathcal{O}(\exp(i\rho(\frac{1}{2}\delta \sin \Theta_o)))\exp(-i\rho\xi_i)$$

$$= \mathcal{O}(\exp(-\delta/\varepsilon))|\exp(-\rho\xi)| = o(\rho\xi)^{-1}\exp(-\rho\xi) .$$

This shows that the second of the four error terms of (4.2) is smaller than the first term of (4.2).

4.4. The third of the error terms in (4.2) if readily estimated by repeating the computation of (4.21):

$$(4.22) \quad \int_{\Lambda_3} |\exp(-\rho s)s^{-1}ds| \leq |\exp(-\rho\tilde{\xi})|\delta^{-1}\int_{\Lambda_3} |ds|$$

$$\leq \exp(-i\rho\hat{\xi}_i)\delta^{-1}(\pi\delta/4) = \frac{1}{2}\pi \exp(-i\rho\hat{\xi}_i)$$

$$= o(\rho\xi)^{-1}\exp(-\rho\xi) .$$

The last of the error terms in (4.2) is the reason for our assumption that $\varphi(\xi) = \mathcal{O}(e^{\kappa|\xi|})$ as $\xi \to -i\infty$ (Chapter 2). The other end of Λ_4 is $\xi_b = \delta\exp(-\pi i/4)$; there, $\varphi(\xi) = \mathcal{O}(1/\delta)$ by assumption (A). Accordingly,

$$|_{\Lambda_4}|\exp(-\rho s)(\varphi(s) - \gamma/s)ds| = \mathcal{O}(1/\delta)\int_{\Lambda_4}|\exp(\kappa|s| - \rho s)ds|$$

$$= \mathcal{O}(1/\delta)\int_{\delta}^{\infty}\exp((\kappa + i\rho)t)dt = \mathcal{O}(1/\delta)(-\kappa + 2/\varepsilon)^{-1}\exp((\kappa - 2/\varepsilon)\delta)$$

$$= \mathcal{O}(\varepsilon/\delta)\exp(-2\delta/\varepsilon).$$

But the ξ which we are considering here have positive imaginary parts, so that $\rho\xi$ has negative (or zero) real part, so

$$\exp(-2\delta/\varepsilon) = \mathcal{O}(\exp(-\rho\xi))\exp(-2\delta/\varepsilon) = o(\exp(-\rho\xi)).$$

Therefore,

$$(4.23)\quad \int_{\Lambda_4}|\exp(-\rho s)(\varphi(s) - \gamma/s)ds| = (\rho\xi)^{-1}\exp(-\rho\xi)\mathcal{O}(\exp(-2\delta/\varepsilon))$$

$$= o(\rho\xi)^{-1}\exp(-\rho\xi).$$

Now substitute into equation (4.2) the equation for its dominant term, (4.3), and the equations for its four error terms, (4.12), (4.21), (4.22), and (4.23). The result is:

$$(4.24)\quad j(\xi)e^{-\rho\xi} = \int_{\xi}^{-i\infty} e^{-\rho s}(\gamma/s)ds + \mathcal{O}(g(\delta)(\rho\xi)^{-1}e^{-\rho\xi}$$

$$+ \mathcal{O}(\exp(-2\delta/\varepsilon))(\rho\xi)^{-1}e^{-\rho\xi}$$

$$= (1 + \mathcal{O}(h(\varepsilon))) \int_{\xi}^{-i\infty} e^{-\rho s}(\gamma/s)ds$$

$$= (\gamma + o(1))(\rho\xi)^{-1}e^{-\rho\xi} .$$

This holds uniformly in ξ on the arc $|\xi| = \delta$, $0 \le \arg\xi \le \pi - \theta_0$ (Figure 3.1 (a)).

Substituting this into (2.12) gives an equation for $j(\xi, s)$

$$(4.25)\qquad j(\xi, s)e^{-\rho s} = (1 + \mathcal{O}(h(\varepsilon))j^{\gamma}(\xi,s)e^{-\rho s}$$

$$= (\gamma + o(1))[(\rho s)^{-1}e^{-\rho s} - (\rho\xi)^{-1}e^{-\rho\xi}] .$$

This holds uniformly in ξ and s on the arc $|\xi| = |s| = \delta$, with $0 \le \arg\xi \le \pi - \Theta_0$, $0 \le \arg s \le \pi - \Theta_0$ (Figure 3.1 (a)).

4.5. We still lack an estimate for $j(\xi,s)$ with ξ or s near ξ_ℓ, where (4.3) fails because ξ_ℓ is a sector boundary of that approximation. By (2.12),

$$(4.26)\qquad j(\xi_r, \xi_\ell)\exp(-\rho\xi_\ell) = \int_{\xi_\ell}^{\xi_r} e^{-\rho s}\varphi(s)ds .$$

and by (A),

$$\varphi(s) = (\gamma/s) + (\tilde{g}(s)/s) . \tag{4.27}$$

with $\tilde{g}(s) \to 0$ as $|s| \to 0$.

In the lower half s-plane, ρs has positive real part, and the exponential is small, but φ has a possible branch cut along the negative imaginary axis (Figure 2.1). Write the upper semicircle (Figure 4.2 (a)) as the sum of curves L' and Γ (Figure 4.2 (b)). Then

$$\jmath(\xi_r, \xi_\ell) \exp(-\rho\xi_\ell) = \gamma \int_{L'} e^{-\rho s} s^{-1} ds + \int_\Gamma e^{-\rho s} \varphi(s) ds + \int_{L'} e^{-\rho s} \tilde{g}(s) s^{-1} ds . \tag{4.28}$$

The integrand of the first integral in (4.28) has only a simple pole,

$$\gamma \int_{L'} e^{-\rho s} s^{-1} ds = -2\pi i \gamma . \tag{4.29}$$

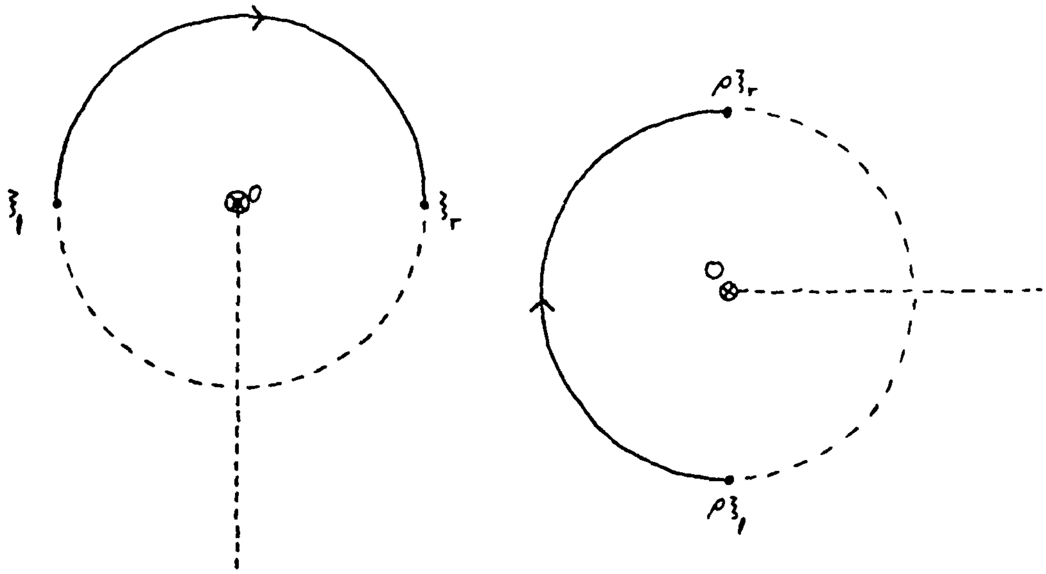
We shall show that this is the dominant term of (4.28); the second term has a decaying exponential and the third term includes $\tilde{g}(s)$, which is small.

The second term of (4.28) would normally be shown small by Jordan's lemma, but the standard forms of Jordan's lemma do not cover this case. Break Γ into three parts, as shown in Figure 4.2 (e). Because φ is holomorphic, assumption (A) implies that on Γ_1,

FIGURE 4.2

Paths for computing $j(\xi_l, \xi_r)$

(a) : original path



(b) : L'

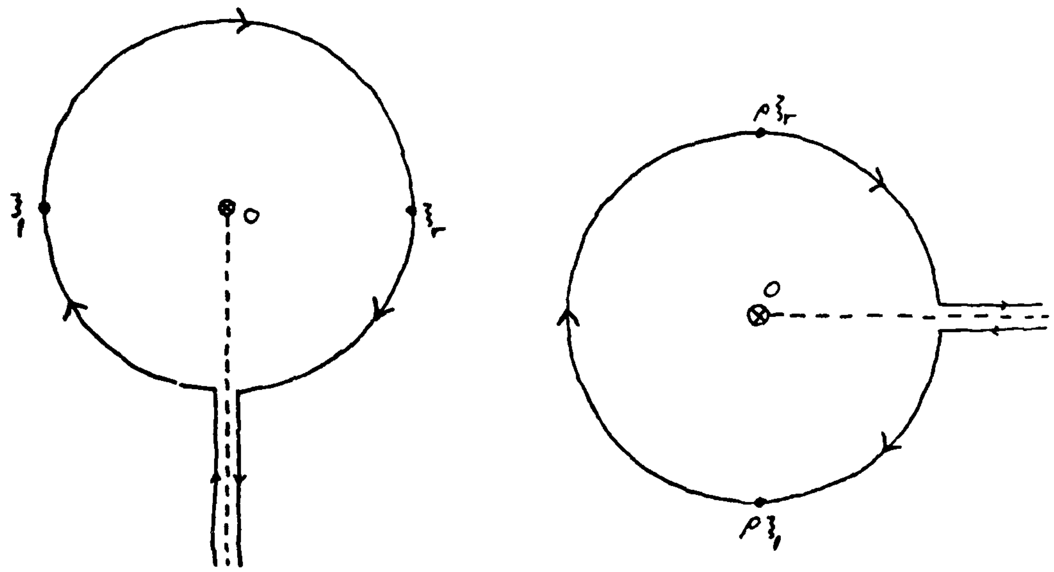
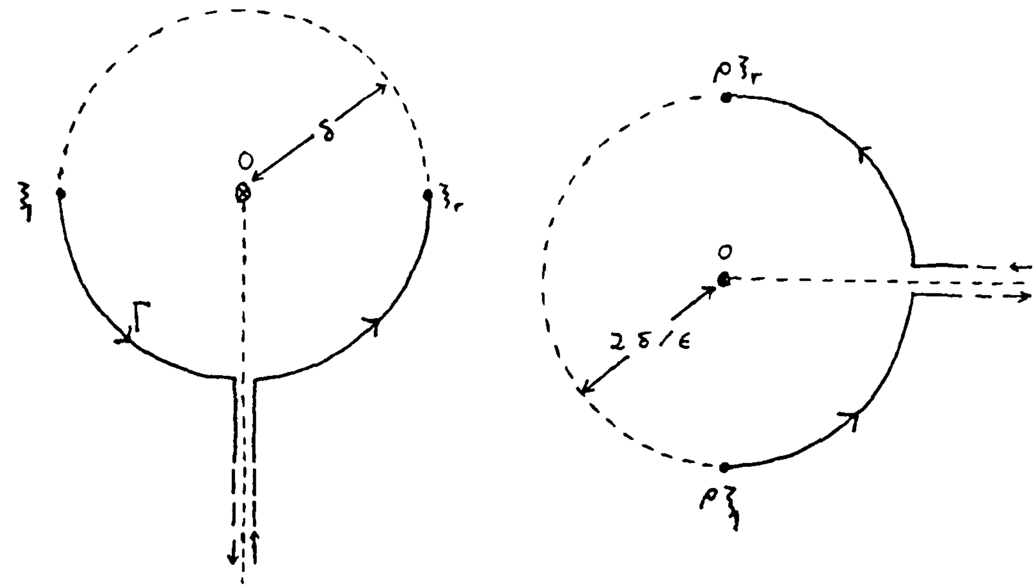


FIGURE 4.2

Paths for computing $j(\xi_l, \xi_r)$

(c) : Γ



(d) : L

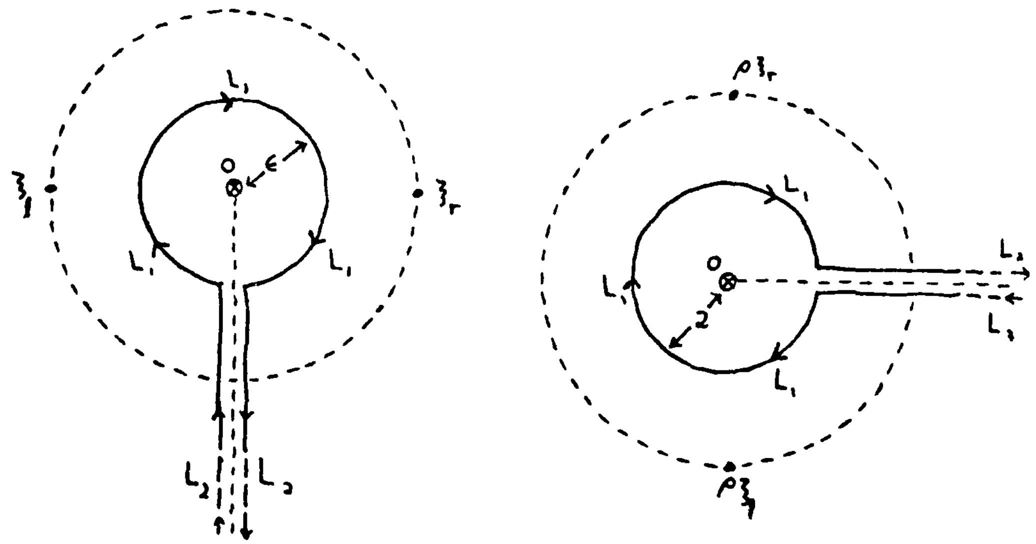
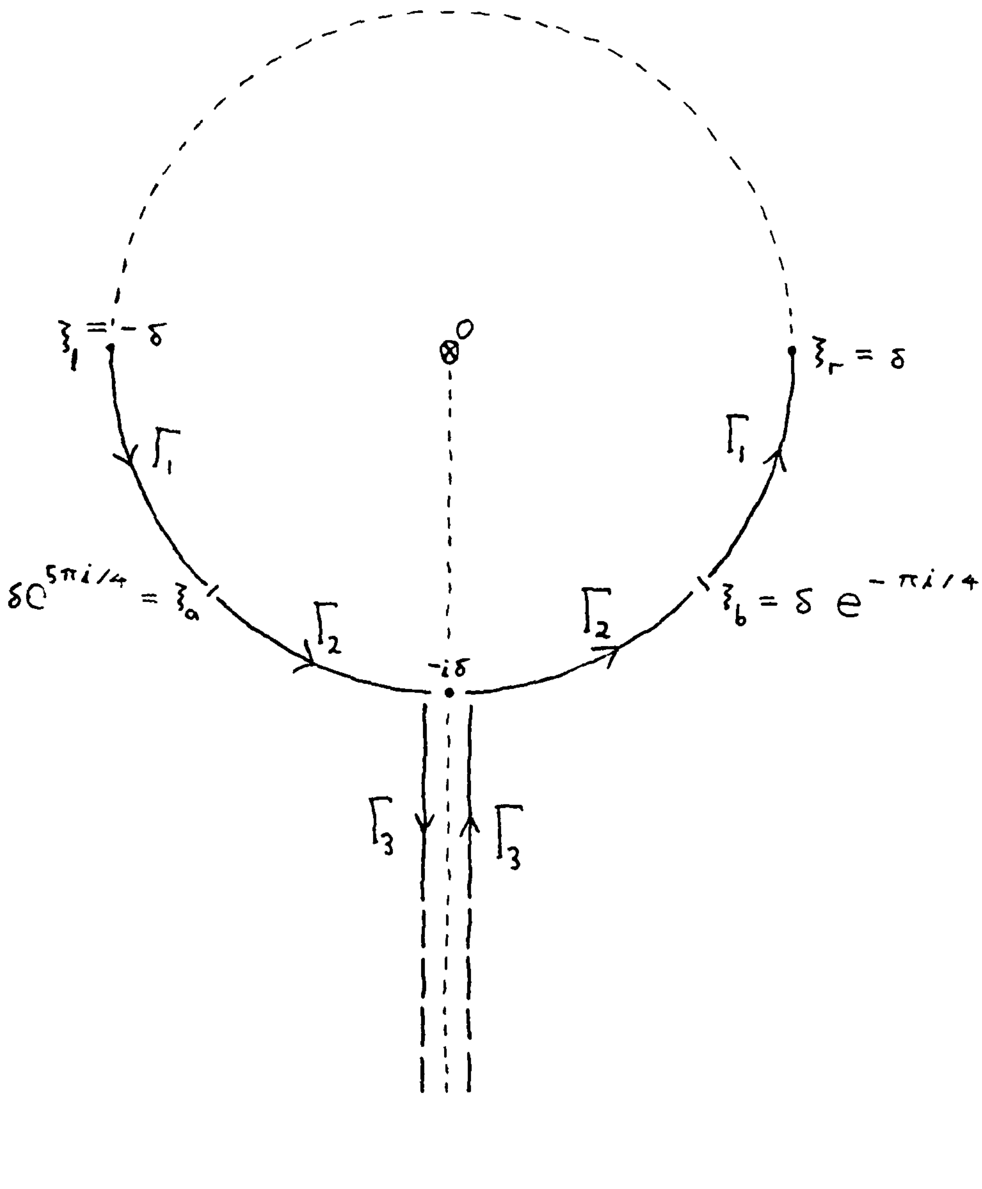


FIGURE [illegible].2

Paths for computing $j(\xi_l, \xi_r)$

(e) : $\Gamma_1, \Gamma_2, \Gamma_3$



$$d\varphi(s)/ds = \mathcal{O}(s^{-2}) = \mathcal{O}(\delta^{-2}) \quad . \tag{4.30}$$

Therefore we can get a useful error bound by integrating by parts.

$$\int_{\Gamma_1} e^{-\rho s}\varphi(s)ds = -\rho^{-1}e^{-\rho s}\varphi(s)\Big|_{\xi_\ell}^{\xi_a} - \rho^{-1}e^{-\rho s}\varphi(s)\Big|_{\xi_b}^{\xi_r}$$

$$+ \rho^{-1}\int_{\Gamma_1} e^{-\rho s}\varphi'(s)ds \tag{4.31}$$

$$= \mathcal{O}(\varepsilon/\delta) \ .$$

For the second part,

$$\mathrm{Im}(s) \le -\delta/\sqrt{2}$$

so that

$$\int_{\Gamma_2} e^{-\rho s}\varphi(s)ds = \mathcal{O}(\exp(-\sqrt{2}(\delta/\varepsilon)))\int_{\Gamma_2}|\varphi(s)ds| = o(\varepsilon/\delta) \ . \tag{4.32}$$

For the third part, we use the assumption that φ grows at most exponentially as $s \to -i\infty$, and assumption (A) for $s \to 0$.

$$\left|\int_{\Gamma_3} e^{-\rho s}\varphi(s)ds\right| \le 2\delta^{-1}\int_{\delta}^{\infty}\exp((\kappa - 2/\varepsilon)t)dt = \mathcal{O}(\varepsilon/\delta) \ . \tag{4.33}$$

Combining (4.31), (4.32), and (4.33) gives a bound for the second term of (4.28)

$$(4.34) \qquad \int_{\Gamma} e^{-\rho s}\varphi(s)\,ds = \mathcal{O}(\varepsilon/\delta)\ .$$

In the third term of (4.28), the exponential factor in the integrand can grow. We shrink the circle in L' from a radius δ to a radius ε (Figure 4.2 (b) and (d)); then on the resulting new curve L, the exponential satisfies

$$(4.35) \qquad |\exp(-\rho s)| \leq \exp(-\rho i\varepsilon) = e^{2}\ .$$

Thus the third term of (4.28) is, where L_1 is the circular part of L and L_2, the straight part,

$$(4.36) \qquad \int_{L'} e^{-\rho s}\tilde{g}(s)s^{-1}ds = \int_{L} e^{-\rho s}\tilde{g}(s)s^{-1}ds$$

$$= \int_{L_1} e^{-\rho s}\tilde{g}(s)s^{-1}ds + \int_{L_2} e^{-\rho s}\tilde{g}(s)s^{-1}ds$$

$$= \mathcal{O}(g(\varepsilon)) + \int_{L_2} e^{-\rho s}\tilde{g}(s)\,s^{-1}ds\ .$$

In I_2, the last integral of (4.36), $\tilde{g}(s)$ is small where $\exp(-\rho s)$ is biggest. However, standard forms of Watson's Lemma do not apply.

Let $g_+(s)$ represent the value of $g(s)$ on the right edge of the negative imaginary axis and $g_-(s)$ the value on the left. Then,

$$(4.37)\quad I_2 = \int_{-i\varepsilon}^{-i\infty} \exp(-\rho s)(g_-(s) - g_+(s))s^{-1}\,ds\ .$$

With $t = is - \varepsilon$ and $p(t) = g_-(s) - g_+(s)$,

$$(4.38)\qquad I_2 = e^{-2} \int_0^{\infty} \exp(-2t/\varepsilon)(t+\varepsilon)^{-1} p(t)dt\ .$$

Now we break this integral into three parts. First, consider $t \geq 1$. Then because $\varphi(s) = \mathcal{O}(\exp(\kappa|s|))$ for $s \to -i\infty$, also $p(t) = \mathcal{O}(\exp(\kappa t))$. Therefore,

$$(4.39)\qquad \begin{aligned} \int_1^{\infty} \exp(-2t/\varepsilon)(t+\varepsilon)^{-1} p(t)dt &\leq \int_1^{\infty} \exp((\kappa - 2/\varepsilon)t)dt \\ &= (2/\varepsilon - \kappa)^{-1}\exp(\kappa - 2/\varepsilon) = \mathcal{O}(\varepsilon)\exp(-2/\varepsilon)\ . \end{aligned}$$

Second, consider $\frac{1}{2}\delta \leq t \leq 1$; in this interval,

$$|p(t)| \leq 2g(|t| + \varepsilon) \leq 2g(1+\varepsilon) < 2g(2)\ ,$$

$$(t+\varepsilon)^{-1} \leq (\tfrac{1}{2}\delta + \varepsilon)^{-1} < 2\delta^{-1}\ ,$$

$$(4.40)\qquad \begin{aligned} \int_{\frac{1}{2}\delta}^{1} \exp(-2t/\varepsilon)(t+\varepsilon)^{-1} p(t)dt &\leq \mathcal{O}(\delta)^{-1} \int_{\delta}^{1} \exp(-2t/\varepsilon)dt \\ &= \mathcal{O}(\varepsilon/\delta)\exp(-2\delta/\varepsilon)\ . \end{aligned}$$

Lastly, consider $0 \leq t \leq \frac{1}{2}\delta$. Here,

$$|p(t)| \leq 2g(|t| + \varepsilon) \leq 2g(\tfrac{1}{2}\delta + \varepsilon) = \mathcal{O}(g(\delta)),$$

$$\int_0^{\frac{1}{2}\delta} \exp(-2t/\varepsilon)(t+\varepsilon)^{-1} p(t)dt \leq \mathcal{O}(g(\delta))\varepsilon^{-1} \int_0^{\frac{1}{2}\delta} \exp(-2t/\varepsilon)dt$$

(4.41)

$$= \mathcal{O}(g(\delta)) .$$

Finally combine (4.39), (4.40), and (4.41) to bound I_2:

(4.42) $$I_2 = \mathcal{O}(g(\delta)) + \mathcal{O}(\varepsilon/\delta)\exp(-2\delta/\varepsilon) = o(1) .$$

From (4.42), (4.36), (4.34), (4.29) and (4.28),

(4.43) $$j(\xi_r, \xi_\ell)\exp(-\rho\xi_\ell) = -2\pi i\gamma + \mathcal{O}(g(\delta)) + \mathcal{O}(\varepsilon/\delta)$$

$$= -2\pi i\gamma + o(1) .$$

4.6. Besides computing $j(\xi_r, \xi_\ell)$, we must find out how close it is to $j^\gamma(\xi_r, \xi_\ell)$. By (4.28) and (3.14),

(4.44) $$(j(\xi_r, \xi_\ell) - j^\gamma(\xi_r, \xi_\ell))\exp(-\rho\xi_\ell) = \int_\Gamma e^{-\rho s}\tilde{g}(s)s^{-1}ds$$

$$+ \int_{L'} e^{-\rho s}\tilde{g}(s)s^{-1}ds .$$

By (4.36) and (4.42), the second term is

$$(4.45)\quad \int_{L'} e^{-\rho s}\tilde{g}(s)s^{-1}ds = \mathcal{O}(g(\varepsilon)) + \mathcal{O}(g(\delta)) + \mathcal{O}(\varepsilon/\delta)\exp(-2\delta/\varepsilon)$$

$$= \mathcal{O}(h(\varepsilon)) .$$

For the first term of (4.44), we use equations (4.31), (4.32), and (4.33) with $\tilde{g}(s)s^{-1}$ in place of $\varphi(s)$. Because $\tilde{g}$ is holomorphic, on Γ_1,

$$(4.46)\qquad d\tilde{g}(s)/ds = \mathcal{O}(g(\delta))/\delta$$

$$\frac{d}{ds}\frac{\tilde{g}(s)}{s} = \mathcal{O}(g(\delta)\delta^{-2}) .$$

Substitute this into (4.31) to find that

$$(4.47)\qquad \int_{\Gamma_1} e^{-\rho s}\tilde{g}(s)s^{-1}ds = \mathcal{O}(\varepsilon\delta^{-1}g(\delta)) .$$

If $\tilde{g}$ is substituted for φ in (4.32) and (4.33), we get

$$(4.48)\qquad \int_{\Gamma_2} e^{-\rho s}\tilde{g}(s)s^{-1}ds = o(\varepsilon\delta^{-1}g(\delta))$$

$$(4.49)\qquad \int_{\Gamma_3} e^{-\rho s}\tilde{g}(s)s^{-1}ds = \mathcal{O}(\varepsilon\delta^{-1}g(\delta)) .$$

Therefore,

$$(4.50)\quad (j(\xi_r, \xi_\ell) - j^\gamma(\xi_r, \xi_\ell))\exp(-\rho\xi_\ell) = \mathcal{O}(h(\varepsilon)) + \mathcal{O}(\varepsilon\delta^{-1}g(\delta))$$

$$= \mathcal{O}(h(\varepsilon)) = \mathcal{O}(h(\varepsilon))j^\gamma(\xi_r, \xi_\ell)\exp(-\rho\xi_\ell)) .$$

4.7. Equation (4.43) leads to a more general equation. Add $j(\xi, \xi_r)$ and use (2.12) and (4.25),

$$(4.51)\quad j(\xi, \xi_\ell)\exp(-\rho\xi_\ell) = (\gamma + o(1))[(\rho\xi_r)^{-1}\exp(-\rho\xi_r)$$

$$- (\rho\xi)^{-1}\exp(-\rho\xi) - 2\pi i]$$

as $\varepsilon \to 0$, uniformly in ξ on the arc $|\xi| = \delta$, $0 \le \arg\xi \le \pi - \Theta_o$ (Figure 3.1 (a)).

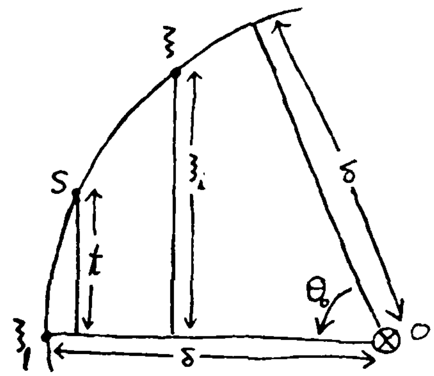
This has still not dealt with $j(\xi, s)$ for ξ and s both near ξ_ℓ. Next we shall find a bound for $j(\xi, \xi_\ell)$ for ξ near ξ_ℓ; i.e., $\pi - \Theta_o \le \arg\xi \le \pi$. By assumption (A), $\varphi(s) = \mathcal{O}(s^{-1})$ uniformly. Therefore, if we perform the integration (2.12) on an arc of the circle $|s| = \delta$ and set $t = \mathrm{Im}(s)$ (Figure 4.3),

$$(4.52)\quad j(\xi, \xi_\ell)\exp(-\rho\xi_\ell) \le \int_{\xi_\ell}^{\xi} |\exp(-\rho s)\varphi(s)ds|$$

$$= \mathcal{O}(1)\int_{\xi_\ell}^{\xi} |\exp(-\rho s)s^{-1}ds|$$

$$= \mathcal{O}(1)\int_0^{\xi_1} \exp(-i\rho t)(\delta^2 - t^2)^{-\frac{1}{2}}dt .$$

FIGURE 4.3:
Variables in
equation (4.52)



Now, because $\pi - \Theta_o \leq \arg s \leq \pi$, we can bound $(\delta^2 - t^2)$ as follows:

$$0 \leq t = \mathrm{Im}(s) = \delta \sin \arg s = \delta \sin(\pi - \arg s) \leq \delta \sin \Theta_o$$

(4.53) $$\delta^2 \geq (\delta^2 - t^2) \geq \delta^2(1 - \sin^2 \Theta_o) .$$

Substitute this into (4.52).

(4.54) $$j(\xi, \xi_\ell) \exp(-\rho\xi_\ell) = \mathcal{O}(\delta^{-1}) \int_0^{\xi_i} \exp(-i\rho t) dt$$

$$= \mathcal{O}(\varepsilon/\delta) \exp(-i\rho\xi_i) = \mathcal{O}(\rho\xi)^{-1} \exp(-\rho\xi)$$

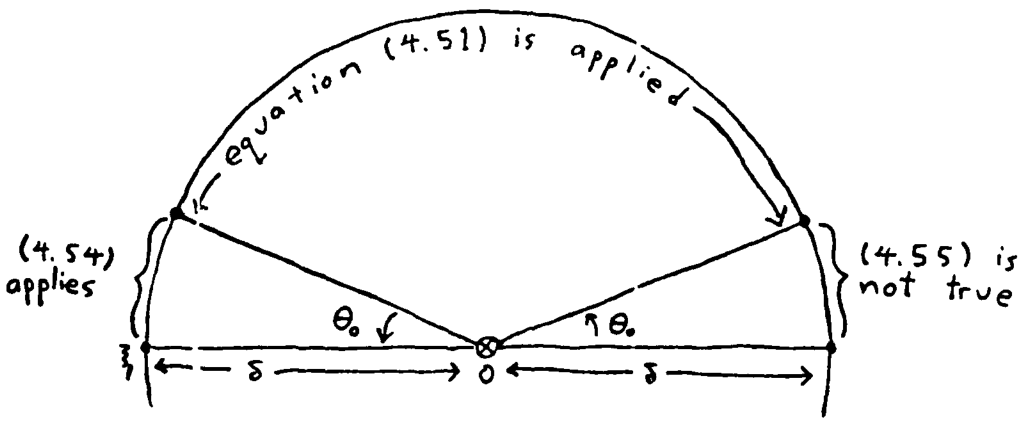
But by (4.51) this bound also applies to any ξ on the arc $|\xi| = \delta$, $\Theta_o \leq \arg\xi \leq \pi - \Theta_o$ (Figure 4.4). Consequently,

(4.55) $$j(\xi, \xi_\ell) \exp(-\rho\xi_\ell) = \mathcal{O}(\rho\xi)^{-1} \exp(-\rho\xi) \qquad \text{as} \quad \varepsilon \to 0$$

uniformly for ξ on the arc $|\xi| = \delta$, $\Theta_o \leq \arg\xi \leq \pi$ (Figure 4.4). Now use this equation twice, to subtract $j(s, \xi_\ell) \exp(-\rho\xi_\ell)$ from $j(\xi, \xi_\ell) \exp(-\rho\xi_\ell)$. By (2.12), for ξ and s as in (4.55),

(4.56) $$j(\xi, s) \exp(-\rho s) = \mathcal{O}(\rho\xi)^{-1} \exp(-\rho\xi) + \mathcal{O}(\rho s)^{-1} \exp(-\rho s) .$$

FIGURE 4.4 : Derivation of equation (4.55)



These are both increasing exponentials in (4.56). If we restrict ξ and s to the left half of the semicricle with $\frac{1}{2}\pi \le \arg\xi \le \arg s \le \pi$ (Figure 3.1 (c)), then the first term dominates, so (4.56) reduces to

$$(4.57) \quad j(\xi, s)\exp(-\rho s) = \mathcal{O}(\rho\xi)^{-1}\exp(-\rho\xi) \qquad \text{as} \quad \varepsilon \to 0 .$$

4.8. To complete the proof of the estimates (3.17), (3.18) and (3.19), which were essential to Section 3, observe that (3.17) is the same as (4.25). For equation (3.18), add (4.25) with $s = \xi_r$ to (4.50).

To get equation (3.19), we must modify the computations in section 4.7. First, subtract (2.12) from itself as follows.

$$(4.59) \quad [j(\xi,\xi_\ell) - j^{\gamma}(\xi,\xi_\ell)]\exp(-\rho\xi_\ell) = \int_{\xi_\ell}^{\xi} \exp(-\rho s)\tilde{g}(s)s^{-1}ds$$

$$\le \mathcal{O}(g(\delta)) \int_{\xi_\ell}^{\xi} |\exp(-\rho s)s^{-1}ds|$$

The last integral also appears in (4.52) and was bounded in (4.54), by

$$(4.60) \qquad \mathcal{O}(\rho\xi)^{-1}\exp(-\rho\xi) = \mathcal{O}(\varepsilon\delta^{-1})\exp(-\rho\xi) .$$

Therefore,

$$(4.61)\quad [j(\xi,\xi_\ell) - j^\gamma(\xi,\xi_\ell)]\exp(-\rho\xi_\ell) = \mathcal{O}(g)\ \varepsilon\ \delta^{-1}\exp(-\rho\xi)\ .$$

Again subtract (2.12) from itself, and apply (4.61) to the result.

$$[j(\xi,s) - j^\gamma(\xi,s)]\exp(-\rho s)$$

$$(4.62)\quad = [j(\xi,\xi_\ell) - j^\gamma(\xi,\xi_\ell)]\exp(-\rho\xi_\ell) - [j(s,\xi_\ell) - j^\gamma(s,\xi_\ell)]\exp(-\rho\xi_\ell)$$

$$= \mathcal{O}(g)\ \varepsilon\ \delta^{-1}\exp(-\rho\xi) + \mathcal{O}(g)\ \varepsilon\ \delta^{-1}\exp(-\rho s)$$

But equation (3.19) is required to hold only for $\frac{1}{2}\pi \le \arg\xi \le \arg s \le \pi$ (Figure 3.1 c); there, $\exp(-\rho s) \le \exp(-\rho\xi)$. Thus, (4.62) reduces to

$$(4.63)\quad [j(\xi,s) - j^\gamma(\xi,s)]\exp(-\rho s) = \mathcal{O}(g)\ \varepsilon\ \delta^{-1}\exp(-\rho\xi)$$

which is equation (3.19).

4.9. In Section 6, where connections are made for ξ on the real line, we will need a bound on $j(\xi,s)$ for ξ and s real, with the same sign. Since $\varphi(s)$ is holomorphic and $\varphi(s) = \mathcal{O}(s^{-1})$ as $|s| \to 0$, for all s not on the negative imaginary axis, the bound $d\varphi(s)/ds = \mathcal{O}(s^{-2})$ holds for real $s \to 0$. Therefore we can integrate (2.12) by parts:

$$
\begin{aligned}
j(\xi, s)e^{-\rho s} &= -\rho^{-1} e^{-\rho t} \varphi(t)\Big|_s^{\xi} + \rho^{-1} \int_s^{\xi} e^{-\rho t} \varphi'(t)\,dt \\
&= \mathcal{O}(\varepsilon t^{-1}) \Big|_s^{\xi} + \mathcal{O}(\varepsilon) \int_s^{\xi} t^{-2}\,dt \\
&= \mathcal{O}(\varepsilon t^{-1}) \Big|_s^{\xi}
\end{aligned}
$$

If $\delta_i \le |\xi|, |s| \le \delta_{i+1}$, then this implies that

$$
j(\xi, s) = \mathcal{O}(\varepsilon \delta_i^{-1}) \quad . \tag{4.65}
$$

5. <u>Completion of Close Connection</u>.

5.1. We shall compute $\varphi(\xi)$ when $q(z) = z^{\nu} \log^{\mu} z$ $(\nu > -1)$ near $z = 0$ to show that assumption (A) of Section 3 holds for such $q(z)$. In this subsection, we use the notation $\log^{\mu} z = (\log z)^{\mu}$. and asymptotic relations hold as $z \to 0$; i.e., $\xi \to 0$.

$$(5.1) \qquad q'(z) = z^{\nu-1} \log^{\mu} z \, (\nu + \mu \log^{-1} z)$$

$$= z^{-1} q(z)(\nu + \mu \log^{-1} z) = z^{-1} q(z)(\nu + o(1)) .$$

Substitute this into the definition (2.7) of $\varphi(\xi)$.

$$(5.2) \qquad \varphi = \frac{1}{2} q^{-2} q' = \frac{1}{2} z^{-1} q^{-1} (\nu + \mu \log^{-1} z)$$

$$= \frac{1}{2} (zq)^{-1} (\nu + o(1)) .$$

To compare this with assumption (A) we must write zq in terms of ξ. Integrate the definition (2.3) of ξ by parts and use (5.1),

$$(5.3) \qquad \xi = zq - \int_0^z q' z \, dz =$$

$$= zq - \nu\xi - O(\log^{-1} z) \int_0^z |q(z') dz'|$$

To see that the last integral in (5.3) is of the same order as $|\xi|$, set $z = r \exp(i\alpha)$ with $0 < r < 1$ and write $|\xi|$ as:

$$(5.4) \qquad |\xi| = \left| \int_0^r s^\nu (\log s + i\alpha)^\mu \, ds = \right| \int_0^r s^\nu \log^\mu s \, ds \Big|$$

$$+ \mathcal{O}(\log^{-1} r) \int_0^r | s^\nu \log^\mu s \, ds |$$

$$= (1 + \mathcal{O}(\log^{-1} r)) \int_0^r | s^\nu \log^\mu s \, ds |$$

$$= (1 + \mathcal{O}(\log^{-1} r)) \int_0^z | q(z') \, dz' | \quad .$$

Therefore, (5.3) says that

$$(5.5) \qquad zq = \xi (1 + \nu + \mathcal{O}(\log^{-1} z)) = \xi (1 + \nu + o(1)) \ .$$

Now (5.2) and (5.5) imply assumption (A):

$$(5.6) \qquad \varphi = \frac{1}{2} (1 + \nu)^{-1} \nu \, \xi^{-1} (1 + \mathcal{O}(\log^{-1} z))$$

$$= \frac{1}{2} (1 + \nu)^{-1} \nu \, \xi^{-1} (1 + o(1)) \ .$$

With (5.5) and the definition of $g(z)$, this also gives us $g(\xi)$:

$$\log \xi + o(1) = \log z + \log q = (1 + \nu + o(1)) \log z$$

$$(5.7) \qquad g(\delta) = \mathcal{O}(\log^{-1} z) = \mathcal{O}(\log^{-1} \xi) = \mathcal{O}(\log^{-1} \delta) \quad ,$$

5.2. We need the connection formulae for A^{γ}, that is, for

$$(5.8) \qquad q(z) = z^{\nu} \qquad (\nu > -1),$$

we must obtain the equations

$$(5.9) \qquad \begin{aligned} &(a) \quad A^{\gamma}(1) = A^{\gamma}(\delta) + o(1) \\ &(b) \quad A^{\gamma}(-1) = A^{\gamma}(-\delta) + o(1) \\ &(c) \quad A^{\gamma}(1) = A^{\gamma}(-1) - B^{\gamma}(-1)\, 2i \sin(\gamma\pi) + o(1) \end{aligned}$$

where

$$(5.10) \qquad \gamma = \frac{1}{2}\, \nu\, (\nu + 1)^{-1} \qquad (-\infty < \gamma < \tfrac{1}{2}) \quad .$$

When (5.8) holds, the solution $w(z)$ to (2.1) is explicitly, in terms of the Hankel functions $H_{\lambda}^{(1)}$ and $H_{\lambda}^{(2)}$,

$$(5.11) \qquad w = C_1 \xi^{\lambda} H_{\lambda}^{(1)}(\xi/\varepsilon) + C_2 \xi^{\lambda} H_{\lambda}^{(2)}(\xi/\varepsilon)$$

where C_1 and C_2 are arbitrary constants and

$$(5.12) \qquad \lambda = \frac{1}{2}(\nu + 1)^{-1} = \frac{1}{2} - \gamma \quad .$$

The connection formulae will follow from (5.11), (2.2) and the appropriate formulae for the Hankel functions. It is known [Olver, 1974, pp. 238-239] that

(5.13) (a) $H_\lambda^{(1)}(\xi/\varepsilon) \sim A_1\, \varepsilon^{1/2}\, \xi^{-1/2} \exp(i\xi/\varepsilon)$

for $-\pi + \theta_0 \le \arg \xi \le \pi$, as $|\xi/\varepsilon| \ge (\delta/\varepsilon) \to \infty$

(b) $H_\lambda^{(2)}(\xi/\varepsilon) \sim A_2\, \varepsilon^{1/2}\, \xi^{-1/2} \exp(-i\xi/\varepsilon)$

for $-\pi \le \arg \xi \le \pi - \theta_0$, as $|\xi/\varepsilon| \ge (\delta/\varepsilon) \to \infty$

(c) $$H_\lambda^{(2)}(\xi/\varepsilon) = H_\lambda^{(1)}(\xi \exp(-\pi i)/\varepsilon) \exp(\lambda \pi i) + H_\lambda^{(2)}(\xi \exp(-\pi i)/\varepsilon)\, 2 \cos(\lambda \pi)$$

with

(5.14) $$A_1 = (2/\pi)^{1/2} \exp\left(-\tfrac{1}{2} i\pi \left(\lambda + \tfrac{1}{2}\right)\right)$$

$$A_2 = (2/\pi)^{1/2} \exp\left(\tfrac{1}{2} i\pi \left(\lambda + \tfrac{1}{2}\right)\right) .$$

We can substitute (5.13) (a) and (b) into (5.11), and this can be compared with (2.2) because

(5.15) $$q^{-1/2} = (\text{constant})\ \xi^{\lambda - 1/2} .$$

by (5.8), and (5.9) (a) follows.

Equation (5.13) (b) does not cover the negative real numbers which (5.9) (b) and (c) need, but we can extend it by using (5.13) (c).

$$H_\lambda^{(2)}(\xi/\varepsilon) \sim 2i A_2 \varepsilon^{1/2} \xi^{-1/2} \cos(\lambda\pi) \exp(i\xi/\varepsilon) + \tag{5.16}$$

$$+ i A_1 \varepsilon^{1/2} \xi^{-1/2} \exp(\lambda\pi i) \exp(-i\xi/\varepsilon) \quad .$$

Now equation (5.9) (b) follows from (5.16), (5.13) (a) and (5.11). For the last part of (5.9) we combine (5.16), (5.13), and (5.11) once more.

$$\begin{aligned} A(1) &= C_1 A_1 \varepsilon^{1/2} \\ A(-1) &= C_1 A_1 \varepsilon^{1/2} + C_2\, 2i A_2\, \varepsilon^{1/2} \cos(\lambda\pi) \\ B(-1) &= C_2\, i A_1\, \varepsilon^{1/2} \exp(\lambda\pi i) \end{aligned} \tag{5.17}$$

Therefore, by (5.17), (5.14), and (5.12),

$$\begin{aligned} A(1) &= A(-1) - C_2\, 2i A_2\, \varepsilon^{1/2} \cos(\lambda\pi) \\ &= A(-1) - B(-1)\,(A_1 \varepsilon^{1/2} \exp(\lambda\pi i))^{-1}\, 2 A_2\, \varepsilon^{1/2} \cos(\lambda\pi) \\ &= A(-1) - B(-1)\, 2i \cos(\lambda\pi) \\ &= A(-1) - B(-1)\, 2i \sin(\gamma\pi) \quad . \end{aligned}$$

5.3. In Section 3.2, we had claimed that the method there used could be carried out more precisely to compute $A(\cdot)$ instead of just bounding it. To do that, take $\xi_d = \xi_\ell$ and $\frac{1}{2}$ [illegible] arg [illegible] $|\xi| = \delta$ (Figure 3.1 (d)) and define

$$I_\varepsilon a(\xi) = \int_{\xi_\ell}^{\xi} a(s)\, j(\xi, s) \exp(\rho\xi - \rho s)\, \varphi(s)\, ds \; .$$

In Section 3.2, we had shown that

$$\|I_\varepsilon a\|_\infty^\ell = \mathcal{O}(\varepsilon/\delta) \; . \tag{5.18}$$

With

$$a_1(\xi) = a(\xi) - a_\ell \exp(\rho\xi - \rho\xi_\ell) \; , \tag{5.19}$$

equation (3.7) may be written as

$$a_1(\xi) = B_\ell\, j(\xi, \xi_\ell) \exp(\rho\xi - \rho\xi_\ell) + I_\varepsilon a(\xi) \tag{5.20}$$

To use (5.18) to contract (5.20), we need $I_\varepsilon a_1(\xi)$ and, in turn, $I_\varepsilon \exp(\rho\xi)$. By (3.4),

$$I_\varepsilon \exp(\rho\xi) = \int_{\xi_\ell}^{\xi} \exp(\rho s)\, j(\xi, s) \exp(\rho\xi - \rho s)\, ds =$$

$$= \mathcal{O}(\rho\xi)^{-1} \int_{\xi_\ell}^{\xi} |\exp(\rho s)\, \varphi(s)\, ds| \; .$$

By (A), $\varphi(s) = \mathcal{O}(\delta^{-1})$. Therefore,

$$I_\varepsilon \exp(\rho\xi) = \mathcal{O}(\delta\rho\xi)^{-1} \int_{\xi_\ell}^{\xi} |\exp(\rho s)\, ds| \; . \tag{5.21}$$

Set $s = \delta \exp(i(\pi - \theta))$ (Figure 5.1) and use the fact that $\sin\theta > \frac{1}{2}\theta$ for $|\theta| \le \frac{1}{2}\pi$ to compute

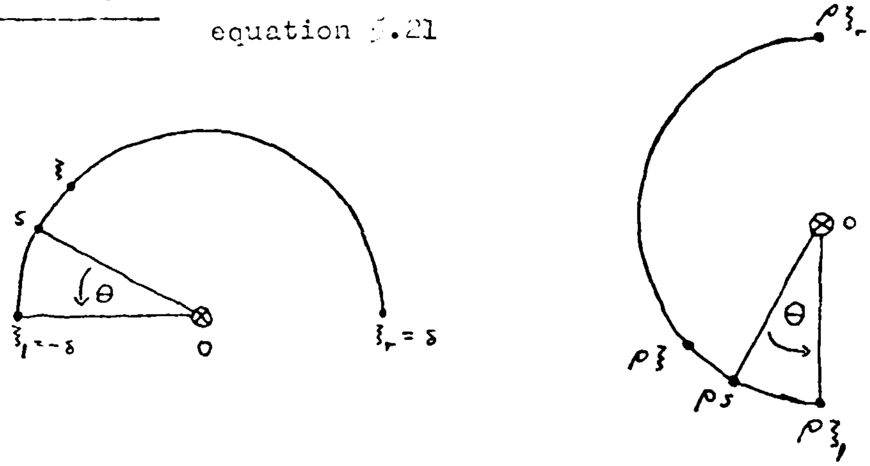
$$\begin{aligned} I_\varepsilon \exp(\rho\xi) &= \mathcal{O}(\varepsilon\delta^{-1}) \int_0^{(\frac{1}{2}\pi - \arg\xi)} \exp(-2\delta\varepsilon^{-1}\sin\theta)\, d\theta \\ &< \mathcal{O}(\varepsilon\delta^{-1}) \int_0^{\frac{1}{2}\pi} \exp(-\delta\varepsilon^{-1}\theta)\, d\theta = \mathcal{O}(\varepsilon/\delta)^2 \; . \end{aligned} \tag{5.22}$$

Therefore, if we apply I_ε to (5.19) we will get

$$\begin{aligned} I_\varepsilon a(\xi) &= I_\varepsilon a_1(\xi) + a_\ell \exp(-\rho\xi_\ell)\, I_\varepsilon \exp(\rho\xi) \\ &= I_\varepsilon a_1(\xi) + \mathcal{O}(\varepsilon/\delta)^2 \; . \end{aligned} \tag{5.23}$$

If we substitute this, (5.18), and (3.4) into (5.20), we get a bound on $a_1(\cdot)$,

FIGURE 5.1 : Variables in equation 5.21



(5.24) $$a_1(\xi) = \mathcal{O}(\varepsilon/\delta) + [\mathcal{O}(\varepsilon/\delta)^2 + I_\varepsilon\, a_1(\xi)]$$
$$= \mathcal{O}(\varepsilon/\delta) + \mathcal{O}(\varepsilon/\delta)\, \|a_1\|_\infty^{\ell} \quad .$$

Therefore,

(5.25) $$a_1(\xi) = \mathcal{O}(\varepsilon/\delta) \quad .$$

This holds uniformly for ξ in the left quarter-circle $\frac{1}{2}\pi \le \arg \xi \le \pi$ (Figure 3.1 (b)). Now substitute (5.25) back into (5.20), using (5.18) again.

(5.26) $$a_1(\xi) = B_\ell\, j(\xi, \xi_\ell) \exp(\rho\xi - \rho\xi_\ell) + \mathcal{O}(\varepsilon/\delta)^2 \quad .$$

If we require $\frac{1}{2}\pi \le \arg \xi \le \pi - \Theta_0$ (Figure 3.1 (d)), then we can use equation (3.3) for $j(\xi, \xi_\ell)$.

(5.27) $$a_1(\xi) = B_\ell(\gamma + o(1))\, [(\rho\xi_r)^{-1} \exp(\rho\xi - \rho\xi_r) - (\rho\xi)^{-1} -$$
$$- 2\pi i \exp(\rho\,\xi)] + \mathcal{O}(\varepsilon/\delta)^2 \quad .$$

But for $\frac{1}{2}\pi \le \arg \xi \le \pi - \Theta_0$,

$$\exp(\rho\xi) = \mathcal{O}(\varepsilon/\delta)^2 \quad ,$$

so the second term in the brackets dominates and we are left with

$$(5.28)\qquad a_1(\xi) = -B_\ell\,(\gamma + o(1))\,(\rho\,\xi)^{-1}\,.$$

Moreover, the first term in (5.19) dominates for exactly the same reason. This gives us the equation we had sought:

$$(5.29)\qquad a(\xi) = -B_\ell(\gamma + o(1))\,(\rho\xi)^{-1} \quad \text{as} \quad \varepsilon \to 0$$

uniformly for $\frac{1}{2}\pi \le \arg \xi \le \pi - \Theta_0$, $|\xi| = \delta$ (Figure 3.1 (d)) .

In terms of the original modulation coefficient $A(\xi)$, (5.25) and (5.29) say that

$$(5.30)\qquad A(\xi) - A_\ell = \Theta\,(\varepsilon\delta^{-1})\exp(-\rho\xi) = \\ = \Theta(\rho\xi)^{-1}\exp(-\rho\xi)$$

uniformly for ξ in the upper-left quarter-circle, $|\xi| = \delta$, $\frac{1}{2}\pi \le \arg \xi \le \pi$ (Figure 3.1 (c)) . Moreover,

$$(5.31)\qquad A(\xi) = -B_\ell(\gamma + O(1))\,(\rho\xi)^{-1}\exp(-\rho\xi)$$

uniformly for ξ in part of the same quarter-circle, the arc $|\xi| = \delta$, $\frac{1}{2}\pi \le \arg \xi \le \pi - \Theta_0$ (Figure 3.1 (d)) .

Since $\exp(-\rho\xi)$ increases as $\operatorname{Im}(\xi)$ increases, this says that $A(\xi)$ increases in magnitude as ξ moves from ξ_ℓ around to $\xi = i\delta$, the halfway point between ξ_ℓ and ξ_r (Figure 3.1 (c)) .

For ξ bounded away from ξ_ℓ by a fixed angle, $A(\xi)$ increases almost exponentially in (δ/ε) as $\varepsilon \to 0$ and, hence, $(\delta/\varepsilon) \to \infty$. However, our choice of $\delta = \delta(\varepsilon)$ may typically have $(\delta/\varepsilon) \sim \log|\log \varepsilon|$, so this "exponential" increase in $A(\xi)$ may actually be rather slow because ξ approaches zero so fast.

For ξ very near to ξ_ℓ, equation (5.31) does not apply, but we can use (5.30). If ξ stay near enough to ξ_ℓ; for example, if the angle between ξ and ξ_ℓ be no more than (ε/δ), then the right-hand side of (5.30) is small, so that $A(\xi)$ is approximately equal to A_ℓ.

6. Connection from Near Zero to Far Away.

6.1. We have obtained formulae connecting the modulation coefficients $A(\xi)$ and $B(\xi)$ at $\xi = \xi_\ell = -\delta$ to those at $\xi = \xi_r = \delta$. But $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, and we might really want to connect points which are much farther from zero, e.g., $\xi = \xi_L = -1$ and $\xi = \xi_R = +1$ or even $\xi = \xi_L = -106\ \varepsilon^{-1}$ and $\xi = \xi_R = \exp(\varepsilon^{-1})$.

The obvious approach to connecting the coefficients at ξ_L to those at ξ_R is to connect ξ_L to ξ_ℓ along the negative real axis, ξ_ℓ to ξ_r by the methods of Sections 2 through 6, and then ξ_r to ξ_R along the positive real axis. The WKB *solution*, $A(\xi) \sim$ constant, $B(\xi) \sim$ constant, is well known to be valid along the negative real axis and along the positive real axis (subject to some integrability and boundedness conditions at $\pm\infty$) so long as ξ is bounded away from the possible singular point $\xi = 0$. This is good enough to connect from points like $\xi_R = \exp(\varepsilon^{-1})$ to fixed points like $\xi_R = 1$, but we still need to push on to $\xi_r = \delta(\varepsilon)$.

This raises the question, how close to zero is the WKB solution really valid? If we could extend it, for real numbers, close enough to zero, then ξ_L and ξ_R would be connected to ξ_ℓ and ξ_r, and the problem would be solved. We shall now show how to carry out such an extension in most cases.

One approach is to imitate the method of Langer (1931). Although he studied only the case where $q(z) \sim z^\nu$ at zero, and obtained an asymptotic formula good at $z = 0$, his method can also be applied to the general case to get a weaker result. For $q(z)$ as was described in Section 2, and for real ξ in R_{ϵ}, it yields

$$(6.1) \quad \begin{aligned} |A(\xi)-A(-1)| &= \mathcal{O}(\varepsilon)\sup\{|(d\phi/d\xi)-2\phi^2|:|\xi|\geq\Delta(\varepsilon)\} \\ |B(\xi)-B(-1)| &= \mathcal{O}(\varepsilon)\sup\{|(d\phi/d\xi)-2\phi^2|:|\xi|\geq\Delta(\varepsilon)\} \end{aligned}$$

uniformly for $\xi \leq -\Delta(\varepsilon)$

$$(6.2) \quad \begin{aligned} |A(\xi)-A(1)| &= \mathcal{O}(\varepsilon)\sup\{|(d\phi/d\xi)-2\phi^2|:|\xi|\geq\Delta(\varepsilon)\} \\ |B(\xi)-B(1)| &= \mathcal{O}(\varepsilon)\sup\{|(d\phi/d\xi)-2\varepsilon^2|:|\xi|\geq\Delta(\varepsilon)\} \end{aligned}$$

uniformly for $\xi \geq \Delta(\varepsilon)$.

Thus, Langer's method can extend the WKB solution to a distance $\Delta(\varepsilon)$ from zero, with $\Delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, provided that the right-hand sides of (6.1), (6.2) approach zero as $\varepsilon \to 0$.

How useful is this? Suppose that assumption (A) of Section 3 holds in addition to the assumptions of Section 2. Then because $\phi = \mathcal{O}(\xi^{-1})$ and ϕ is holomorphic, $(d\phi/d\xi) = \mathcal{O}(\xi^{-2})$ on this real line; therefore,

$$(6.3) \quad (d\phi/d\xi) - 2\phi^2 = \mathcal{O}(\xi^{-2}) \quad \text{as } |\xi| \to 0, \text{ uniformly.}$$

Thus the condition for this extension method to work is

$$\varepsilon\,\Delta(\varepsilon)^{-2} \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{6.4}$$

This allows us to take $\Delta(\varepsilon) = \varepsilon^{\alpha}$ with $\alpha < \frac{1}{2}$, for example.

But such $\Delta(\varepsilon)$ will not be good enough; we must extend the WKB solution still closer to zero. As noted in Section 3.3, a typical interesting $\delta(\varepsilon)$ would have $\delta/\varepsilon = \frac{1}{2}\log|\log\delta|$; while if $\Delta(\varepsilon) = \varepsilon^{\alpha}$ then $\Delta = (2\delta)^{\alpha}(\log|\log\delta|)^{-\alpha} >> \delta$.

6.2. To connect from $\xi = -1$ to ξ_{ℓ} and ξ_r to $\xi = 1$ we will again use the integral equation (2.13). Besides the assumptions of Section 2, we will use a weaker form of assumption (A), Section 3

$$(A') \quad \phi(\xi) = \mathcal{O}(\xi^{-1}) \quad \text{as} \quad |\xi| \to 0, \text{ uniformly.}$$

Since ϕ is holomorphic, it follows that for real ξ,

$$\phi'(\xi) = \mathcal{O}(\xi^{-2}) \quad \text{as} \quad |\xi| \to 0. \tag{6.5}$$

We shall start by connecting between $\pm\delta_1(\varepsilon)$ and $\pm\delta_2(\varepsilon)$, where $\varepsilon \le \delta_1 \le \delta_2$, and

$$\varepsilon\,\delta_1^{-1}\,(\log(\delta_2\delta_1^{-1}))^2 \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{6.6}$$

Equation (6.6) keeps δ_1 and δ_2 from being too far apart for us to control integrals ranging between δ_1 and δ_2.

Write equation (2.13) and its equivalent for positive ξ, equation (3.5), as follows:

$$(6.7) \qquad A(\xi) - A_1 = B_1 \; j(\xi,\xi_1)\exp(-\rho\xi_1) + T_\varepsilon A(\xi)$$

where

$$(6.8) \qquad T_\varepsilon A(\xi) = \int_{\xi_1}^{\xi} \phi(s)\; A(s)\; j(\xi,s)ds$$

and where

$$(6.9) \qquad A_1 = A(\xi_1),\; B_1 = B(\xi_1) \;,\; \xi_1 = \pm\delta_1$$

When $\xi_1 = +\delta_1$, we take $\xi_1 \le \xi \le \delta_2$;

when $\xi_1 = -\delta_1$, then $-\delta_2 \le \xi \le \xi_1$.

We can bound T_ε by means of (A'), (6.5), and (4.65). Let $\|\cdot\|_\infty$ represent the sup norm on the interval over which ξ ranges.

$$|T_\varepsilon A(\xi)| \le \|A\|_\infty \|j\|_\infty \|\xi\phi\|_\infty \int_{\xi_1}^{\xi} |s^{-1}ds|$$

$$(6.10) \qquad \le \|A\|_\infty\, \Theta(\varepsilon/\delta_1)\; \log(\delta_2/\delta_1) = \|A\|_\infty\, o(1)$$

Therefore T_ε is a contraction for sufficiently small ε, and (6.7) has a bounded solution. Substitute (6.10) and (4.65) into (6.7)

$$(6.11) \qquad A(\xi) - A_1 = \Theta(\varepsilon/\delta_1) + \|A\|_\infty\; \Theta(\varepsilon\delta_1^{-1}\log(\delta_2/\delta_1))$$

Therefore $\|A\|_\infty$ is bounded independently of ε, and hence,

$$(6.12) \qquad A(\xi) = A_1 + \mathcal{O}(\varepsilon\delta_1^{-1}\log(\delta_2/\delta_1)) = A_1 + o(1)$$

uniformly for ξ between $\xi_1 = \pm\delta_1$ and $\pm\delta_2$.

To compute the other modulation coefficient, $B(\xi)$, integrate equation (2.8) by parts and use equations (A'), (6.12), (6.5), and (6.6) and the fact that the integral is computed on a real interval.

$$(6.13) \qquad B(\xi) - B_1 = \int_{\xi_1}^{\xi} \phi(s)A(s)e^{\rho s}\,ds =$$

$$= A_1\rho^{-1}e^{\rho s}\phi(s)\Big|_{\xi_1}^{\xi} - A_1\rho^{-1}\int_{\xi_1}^{\xi}\phi'(s)e^{\rho s}\,ds +$$

$$+ \int_{\xi_1}^{\xi}\phi(s)(A(s) - A_1)e^{\rho s}\,ds$$

$$= \mathcal{O}(\varepsilon/\delta_1) + \mathcal{O}(\varepsilon)\int_{\xi_1}^{\xi} s^{-2}ds +$$

$$+ \mathcal{O}(\varepsilon\delta_1^{-1})\log(\delta_2/\delta_1)\int_{\xi_1}^{\xi} s^{-1}ds$$

$$= \mathcal{O}(\varepsilon/\delta_1) + \mathcal{O}(\varepsilon/\delta_1) + \mathcal{O}(\varepsilon\delta_1^{-1})(\log(\delta_2/\delta_1))^2$$

$$= o(1)$$

Thus, for [illegible] between [illegible] and $\pm\delta_2$,

$$B(\xi) = B_{\cdot} + o(1) \tag{6.15}$$

6.4. Our goal is to prove that

$$\begin{aligned} A(\pm\delta) &= A(\pm 1) + o(1) \\ B(\pm\delta) &= B(\pm 1) + o(1) \quad . \end{aligned} \tag{6.15}$$

We cannot use equations (6.12) and (6.14) directly because in Section 6.2, δ_2 and δ_1 must be too close to each other. In the case $q(z) = z^{\nu}(\log z)^{\mu}$, we typically will choose $\varepsilon^{-1}\delta = \frac{1}{2}\log|\log\delta|$. Suppose that we tried to use (6.12) and (6.14) by setting $\delta_1(\varepsilon) = \delta(\varepsilon)$, $\delta_2(\varepsilon) = 1$. Then we would find that

$$\varepsilon\delta_1^{-1}\log^2(\delta_2\delta_1^{-1}) = 2(\log|\log\delta|)^{-1}\log^2\delta \to \infty \tag{6.16}$$

Thus, condition (6.6) would be violated.

Instead, we will choose a finite sequence

$$\delta(\varepsilon) = \delta_1(\varepsilon) \le \delta_2(\varepsilon) \le \cdots \le \delta_n(\varepsilon) = 1$$

for which, for $i = 1,2,\ldots,n-1$,

$$\varepsilon\delta_i^{-1}\log^2(\delta_{i+1}\delta_i^{-1}) \to 0 \quad \text{as} \quad \varepsilon \to 0$$

[illegible] (6.12) and (6.14) do apply, so that our goal, [illegible] immediately.

[illegible] clarity, we begin with an example. Let [illegible], and

(6.19) $$\varepsilon^{-1}\delta = \frac{1}{2}\log|\log\delta|$$

Define δ_1, δ_2, δ_3, δ_4 as follows:

(6.20)
$$\begin{aligned}
&(a)\quad \delta_1 = \delta \\
&(b)\quad \delta_2 = \varepsilon \exp(\log^{(2)}(\varepsilon^{-1}))^{1/4} \\
&(c)\quad \delta_3 = \varepsilon \log^3(\varepsilon^{-1}) \\
&(d)\quad \delta_4 = 1
\end{aligned}$$

Here we are using the notation

$$\log^2 x = (\log x)^2 \quad , \quad \log^{(2)} x = \log\log x$$

or, more generally,

(6.21)
$$\begin{aligned}
\log^{(n+1)} x &= \log\log^{(n)} x \\
\log^{n+1} x &= (\log x)(\log^n x)
\end{aligned}$$

It is obvious that $\delta_1,\dots,\delta_4$ defined by (6.20) satisfy (6.17). To see that they satisfy (6.18) we start by putting ε more explicitly on the right side of (6.19). Divide (6.19) by δ and apply log.

(6.22)
$$\begin{aligned}
\log\varepsilon^{-1} &= \log\delta^{-1} + \log\frac{1}{2} + \log^{(3)}\delta^{-1} = (1+o(1))\log\delta^{-1} \\
\log^{(2)}\varepsilon^{-1} &= o(1) + \log^{(2)}\delta^{-1} = (1+o(1))\log\delta^{-1} \\
\varepsilon^{-1}\delta &= \frac{1}{2}\log^{(2)}\delta^{-1} = (\frac{1}{2} + o(1))\log^{(2)}\varepsilon^{-1}
\end{aligned}$$

Now substitute (6.22) and (6.20 (b)) into (6.18),

$$\varepsilon\delta_1^{-1}\log^2(\delta_2\delta_1^{-1}) = (2+o(1))(\log^{(2)}\varepsilon^{-1})^{-1}[\log(\varepsilon^{-1}\delta_2) - \log(\varepsilon^{-1}\delta_1)]^2 =$$

$$= (2+o(1))(\log^{(2)}\varepsilon^{-1})^{-1}[(\log^{(2)}\varepsilon^{-1})^{1/4} -$$

$$- O(1) - \log^{(3)}\varepsilon^{-1}]^2 =$$

$$= (2+o(1))(\log^{(2)}\varepsilon^{-1})^{-1}[(\log^{(2)}\varepsilon^{-1})^{1/4}]^2 =$$

$$= (2+o(1))(\log^{(2)}\varepsilon^{-1})^{-1/2} \to 0 \quad \text{as} \quad \varepsilon \to 0$$

Thus (6.18) holds for δ_1 and δ_2.

Next substitute (6.20 (b), (c)) into (6.18)

$$\varepsilon\delta_2^{-1}\log^2(\delta_3\delta_2^{-1}) = \exp[-(\log^{(2)}\varepsilon^{-1})^{1/4}][\log(\varepsilon^{-1}\delta_3) - \log(\varepsilon^{-1}\delta_2)]^2$$

$$= \exp[-(\log^{(2)}\varepsilon^{-1})^{1/4}][3\log^{(2)}(\varepsilon^{-1}) - (\log^{(2)}\varepsilon^{-1})^{1/4}]$$

$$= (3+o(1))\exp[-(\log^{(2)}\varepsilon^{-1})^{1/4}]\log^{(2)}\varepsilon^{-1} \to 0$$

$$\text{as } \varepsilon \to 0$$

Thus (6.19) holds for δ_2 and δ_3.

Last, substitute (6.20 (c), (d)) into (6.18).

$$\varepsilon\delta_3^{-1}\log^2(\delta_4\delta_3^{-1}) = \log^{-3}(\varepsilon^{-1})[\log\varepsilon^{-1} - 3\log^{(2)}\varepsilon^{-1}]^2$$

$$= (1+o(1))\log^{-1}\varepsilon^{-1} \to 0 \quad \text{as} \quad \varepsilon \to 0$$

Thus conditions (6.17) and (6.18) hold in their entirety so that (6.15) holds for this example of primary interest.

6.4. We now give another example, one for which we cannot prove (6.15). It indicates that we need a new assumption in order to prove (6.15), and it suggests what form that assumption should take--a slight strenthening of assumption (A).

We shall construct a function $f(\delta)$ which grows to infinity as $\delta \to 0$ more slowly than any iterated logarithm $\log^{(n)}\delta^{-1}$. It is conceivable that some $\emptyset(\xi) = \xi^{-1}(\gamma +$ $+ o(1))$ satisfying (A) might have an error term so large that, to satisfy (3.21), we must have $\varepsilon^{-1}\delta \le f(\delta)$ or $f(\varepsilon)$. We shall show that, for such a case, the method of subsection 6.3 will not work, so we cannot prove (6.15) for (A)

First, we construct $f(\delta)$. For $N = 1,2,3,\ldots$ define

$$f_1(\delta) = \log \delta^{-1}$$

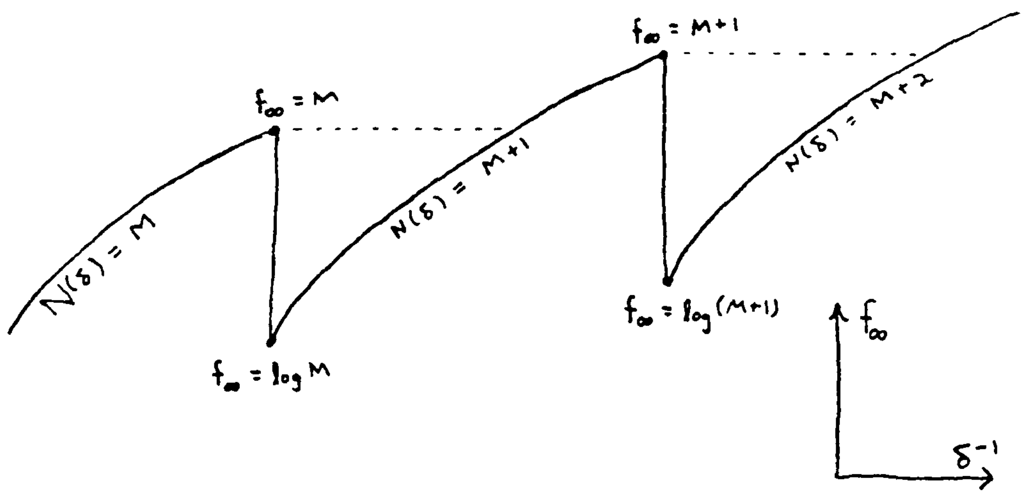
$$f_{N+1}(\delta) = \begin{cases} f_N(\delta) & \text{if } f_N(\delta) \le N \\ \log f_N(\delta) & \text{if } f_N(\delta) > N \end{cases} \tag{6.23}$$

For any given $\delta \in (0,1)$, $\log f_N(\delta) < f_N(\delta)$, so there is a finite $N = N(\delta)$ for which

$$f_{N-1}(\delta) > f_N(\delta) = f_{N+1}(\delta) = \ldots$$

Therefore we may define a limiting function, shown in Figure 6.1.

FIGURE 6.1 : $f_\infty(\delta)$



$$(6.24) \qquad f_\infty(\delta) = \lim_{N\to\infty} f_N(\delta) = f_{N(\delta)}(\delta) = \log^{(N(\delta))} \delta^{-1}$$

Note that $N(\delta)$ increases to infinity as $\delta \to 0$, and that by (6.23), $f_\infty(\delta) \le N(\delta)$. Equation (6.23) also gives a lower bound to f_∞, for if $N = N(\delta)$ then $f_N = \log f_{N-1}$, so that $f_{N-1} > N-1$. Therefore, $f_N > \log(N-1)$. In summary,

$$(6.25) \qquad \log(N(\delta) - 1) < f_\infty(\delta) \le N(\delta) \quad .$$

We make f_∞ monotonic by defining

$$(6.26) \qquad f(\delta) = \sup\{ f_\infty(\delta') : \delta \le \delta' < 1 \}$$

Because $f_\infty(\delta) \to \infty$ as $\delta \to 0$, the same holds true for $f(\delta)$. How fast does $f(\delta)$ grow? Let $\delta \le \delta' < 1$. Then by (6.25),

$$(6.27) \qquad f(\delta') \le N(\delta') \le N(\delta) < 1 + \exp f_\infty(\delta) =$$

$$= 1 + \log^{(N(\delta) - 1)} \delta^{-1}$$

Therefcre

$$(6.28) \qquad f(\delta) \le 1 + \log^{(N(\delta)-1)} \delta^{-1}$$

For any fixed N, (6.28) and the fact that $N(\delta) \to \infty$ imply that

$$(6.29) \qquad f(\delta) \ll \log^{(N)} \varepsilon^{-1} \quad \text{as} \quad \delta \to 0 .$$

With this very slowly growing $f(\delta)$, we define $\delta(\varepsilon)$; first define $\varepsilon(\delta)$ by

$$(6.30) \qquad \varepsilon^{-1}\delta = f(\delta)$$

and then invert to get $\delta(\varepsilon)$ (defined for all small ε except for the jump discontinuities at $\delta^{-1}\varepsilon = 1/f = 1, \frac{1}{2}, 1/3, \ldots$). This $\delta(\varepsilon)$ is extremely close to ε, but it still satisfies (2.14); that is, $\varepsilon \ll \delta \ll 1$.

There is now so much distance between $\delta(\varepsilon)$ and 1 that we can't connect across that distance, using the method of subsection 6.3. Suppose that we try to do so, i.e., that we construct a sequence like (6.17) and (6.18) and try to use it to make a connection. We shall show that our task would never end; that however large N may be, $\delta_N(\varepsilon) \ll 1$ as $\varepsilon \to 0$.

Specifically, we take a sequence $\delta_1, \delta_2, \delta_3, \ldots$ which satisfy (6.17) and (6.18) or even a weakened form of (6.18), that is,

$$(6.31) \qquad \delta(\varepsilon) = \delta_1(\varepsilon) \le \delta_2(\varepsilon) \le \delta_3(\varepsilon) \le \cdots$$

$$(6.32) \qquad \varepsilon\, \delta_i^{-1} \log(\delta_{i+1}\, \delta_i^{-1}) \to 0 \quad \text{as} \quad \varepsilon \to 0 .$$

Then we shall show by induction that the following two equations hold, for all i:

(6.33) $$\delta_i \to 0 \quad \text{as} \quad \varepsilon \to 0$$

(6.34) $$\varepsilon^{-1}\delta_i << \log^{(N)} \delta_i^{-1} \quad \text{as} \quad \varepsilon \to 0, \text{ for each fixed } N.$$

Equation (6.33) is the result we want; (6.34) is a necessary intermediate step in the induction; it says that δ_i can't get vary far from ε.

Equations (6.29) and (6.30) take care of the case $i = 1$. If (6.33) and (6.34) hold for some i, then by (6.32)

(6.35) $$\log(\delta_i^{-1}\, \delta_{i+1}) = o(\delta_i \varepsilon^{-1})$$

Exponentiate this equation to get:

(6.36) $$\delta_{i+1} < \delta_i \exp(\delta_i \varepsilon^{-1})$$

By (6.33) and (6.34) with $N = 2$ this implies that

(6.37) $$\delta_{i+1} < \delta_i \exp(\log^{(2)} \delta_i^{-1}) = \delta_i \log \delta_i^{-1} \to 0 .$$

Now that we've shown (6.33) for $(i+1)$, equation (6.34) remains.

Choose any integer $M \geq 1$, and require ε to be so small that

$$\log^{(M+3)} \delta_i^{-1} > 1$$

By (6.36) and (6.34) with (M+2) and (M+3) in place of N,

$$(6.38) \qquad \varepsilon^{-1}\delta_{i+1} < \varepsilon^{-1}\delta_i \exp(\varepsilon^{-1}\delta_i) <$$

$$< (\log^{(M+2)}\delta_i^{-1}) \exp (\log^{(M+3)}\delta_i^{-1}) =$$

$$= (\log^{(M+2)}\delta_i^{-1})^2 < \log^{(M+1)}\delta_i^{-1}$$

By (6.35) and (6.34) with N = 1.

$$(6.39) \qquad \log \delta_{i+1}^{-1} = \log \delta_i^{-1} + o(\delta_i\varepsilon^{-1}) = (1+o(1))\log \delta_i^{-1}$$

Apply $\log^{(M-1)}$ to equation (6.39)

$$(6.40) \qquad \log^{(M)}\delta_{i+1}^{-1} = (1 + o(1))\log^{(M)}\delta_i^{-1} > \frac{1}{2}\log^{(M)}\delta_i^{-1}$$

$$\gg \log^{(M+1)} \delta_i^{-1}$$

Combining (6.38) and (6.40) gives the result we had sought:

$$(6.41) \qquad \varepsilon^{-1}\delta_{i+1} \ll \log^{(M)} \delta_{i+1}^{-1}$$

6.5. Subsection 6.4 indicates that we will not be able to solve the connection problem fully if $\delta(\cdot)$ be extraordinarily close to ε. To give room for $\delta(\varepsilon)/\varepsilon$ to grow at a reasonable rate, we must have some control over the error term in assumption (A). So we strengthen (A), requiring the error term to approach zero at least as fast as some iterated logarithm:

(A") $\quad \phi(\xi) = \xi^{-1}(\gamma + \mathcal{O}(\log^{(n)}|\xi|^{-1})^{-1})$

uniformly in ξ, as $|\xi| \to 0$, where n is a fixed positive integer and $-\infty < \gamma < \frac{1}{2}$.

This assumption still covers practically all familiar examples; when $q(z) = z^{\nu}\log^{\mu} z$ we may take $n = 1$ in (A").

Provided that (A") holds, we can define $\delta(\varepsilon)$ by:

$$\delta = \varepsilon \log^{(n+2)} \varepsilon^{-1} \tag{6.42}$$

Then for $n \geq 1$,

$$\log \delta = (1 + o(1)) \log \varepsilon \tag{6.43}$$

It is easy to see that this satisfies (2.14). It also satisfies all the other requirements that we have put on $\delta(\varepsilon)$ in earlier sections, for by (A"), (6.43) and (6.42)

$$(6.44) \qquad g(\delta) = \Theta(\log^{(n)} \delta^{-1})^{-1} = \Theta(\log^{(n)} \varepsilon^{-1})^{-1}$$

$$= o(\log^{(n+1)} \varepsilon^{-1})^{-2} = o(\exp(-2\delta/\varepsilon)).$$

We will solve the connection problem by finding $\delta_1, \delta_2, \ldots$ satisfying (6.17) and (6.18). The choice of δ_i will be essentially the same as in the example in 6.3. Set $\delta_1 = \delta(\varepsilon)$. For $i = 1,\ldots,n+1$, set

$$(6.45) \quad \begin{aligned} &\text{(a)} \quad \delta_{2i} = \varepsilon \exp(\log^{(n+3-i)} \varepsilon^{-1})^{1/4} \\ &\text{(b)} \quad \delta_{2i+1} = \varepsilon\, \Theta(\log^{(n+2-i)} \varepsilon^{-1})^{3} \\ &\text{(c)} \quad \delta_{2n+4} = 1 \quad . \end{aligned}$$

We need only verify that (6.18) holds. For the first step, take $i = 1$.

$$(6.46) \quad \varepsilon\delta_1^{-1} \log^2(\delta_2\delta_1^{-1}) = \varepsilon\delta_1^{-1}[\log(\varepsilon\delta_1^{-1}) + \log(\delta_2\varepsilon^{-1})]^2 =$$

$$= (\log^{(n+3)} \varepsilon^{-1})^{-1} \; [-\log^{(n+3)} \varepsilon^{-1} +$$

$$+ (\log^{(n+2)} \varepsilon^{-1})^{1/4} \;]^2$$

$$= (1+o(1)) \; (\log^{(n+2)} \varepsilon^{-1})^{-1/2} \; \to \; 0$$

By almost the same reasoning, we can verify (6.18) for all odd δ_j, except $j = 2n+3$:

$$(6.47)\quad \varepsilon\delta_{2i+1}^{-1} \log^2(\delta_{2i+2}\delta_{2i+1}^{-1}) =$$

$$= (\log^{(n+2-i)}\varepsilon^{-1})^{-3}\ [-3 \log^{(n+3-i)}\varepsilon^{-1} +$$

$$+ (\log^{(n+2-i)}\varepsilon^{-1})^{1/4}]^2 =$$

$$= (1+o(1))\ (\log^{(n+2-i)}\varepsilon^{-1})^{-5/2} \to 0$$

Next we take care of (6.18) for even δ_j:

$$(6.48)\quad \varepsilon\delta_{2i}^{-1} \log^2(\delta_{2i+1}\delta_{2i}^{-1}) = \exp[-(\log^{(n+3-i)}\varepsilon^{-1})^{1/4}][\ldots$$

$$\ldots - (\log^{(n+3-i)}\varepsilon^{-1})^{1/4} + 3 \log^{(n+3-i)}\varepsilon^{-1}]^2 =$$

$$= [(3 + o(1))x]^2 \exp(-x^{1/4}) \to 0$$

where $x = \log^{(n+3-i)}\varepsilon^{-1} \to \infty$

The last step is $\delta_j = \delta_{2n+3}$, i.e., $i = n+1$.

$$\varepsilon\delta_{2n+3}^{-1} \log^2(\delta_{2n+4}\delta_{2n+3}^{-1}) = (\log\varepsilon^{-1})^{-3}[-\log\varepsilon - 3 \log^{(2)}\varepsilon^{-1}]^2$$

$$= (1+o(1))\ (\log \varepsilon^{-1})^{-1} \to 0 \quad .$$

Therefore, (6.18) holds, so (6.15) holds and the connection is complete.

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The solutions of Schroedinger equations

(1.1) $$\varepsilon^2 d^2w/dz^2 + q^2 w = 0, \quad q = q(z),$$

have well-known WKB-approximations, but the coefficients in them differ on the two sides of a turning point. A new method [illegible]

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ABSTRACT (continued)

of turning points, which includes logarithmic branch points of $q(z)$, among many others. To this end, a delicate contraction for an integral equation differing from those of Langer and Olver is used to show that Bessel functions can still approximate the solutions at a certain, small distance from the irregular point of (1.1), even though not uniformly near it. A novel feature of the analysis is that the extreme variation of the exponential kernel is here controlled even on non-progressive paths. Connection is completed radially by means of the same integral equation.

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